INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. These lecture notes correspond to the contents of the Undergraduate course MATH 4220-PDE taught at the Chinese University of Hong Kong in Spring 2024. Partial differential equation (PDE) is a fundamental tool for modeling natural phenomena. The main goal of these notes is to present four classes of linear PDEs (Transport, Laplace, Heat, Wave), to introduce their fundamental properties, and to introduce the mathematical tools that are necessary for their study. Prerequisites for these lecture notes are the bases of Multivariable Calculus (Integration by parts, divergence theorem, Green's Identity, Stokes Formula, Gauss Formula, etc.), of linear algebra, and of mathematical analysis.

1. Introduction

1.1. **What is a PDE.** A *partial differential equation* (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Definition 1.1. Let F be a given function such that

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

An expression of the form

$$F\left(D^k u(x), D^{k-1}u(x), \dots, u(x), x\right) = 0 \quad \text{for all } x \in \Omega \tag{1.1}$$

is called a k-th order partial differential equation, where

$$D^k u(x) = \{\partial_x^\alpha u(x) : |\alpha| = k\} \in \mathbb{R}^{n^k},$$

and $u : \Omega \to \mathbb{R}$ is unknown.

We solve the PDE if we find all $u : \Omega \to \mathbb{R}$ verifying (1.1), possibly only among those functions satisfying certain auxiliary boundary conditions on some part Λ of $\partial\Omega$.

Definition 1.2. An operator $\mathcal{L}$ is called $\mathbb{R}$-linear if it satisfy

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v \quad \mathcal{L}(cu) = c\mathcal{L}u$$

for any functions u and v, and any constant $c \in \mathbb{R}$.

Definition 1.3. The PDE (1.1) is called linear if it has the form

$$\mathcal{L}u(x) = f(x) \quad \text{on } \Omega$$

for given $\mathbb{R}$-linear operator $\mathcal{L}$ and given function f. The linear PDE is homogeneous if $f \equiv 0$ and inhomogeneous if $f \neq 0$.

Example 1.4. (i) Consider the PDE $\partial_t u + \partial_x^3 u + u\partial_x u = 0$ where $u(t,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. The PDE is not linear equation since the operator $\mathcal{L}u = \partial_t u + \partial_x^3 u + u\partial_x u$ is not a linear operator. By direct computation, we check

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v + u\partial_x v + v\partial_x u \Rightarrow \mathcal{L}(u+v) \neq \mathcal{L}u + \mathcal{L}v.$$

(ii) Consider the PDE $\partial_t^2 u - \partial_x^2 u + u^3 = 0$ where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The PDE is not linear equation since the operator $\mathcal{L}u = \partial_t^2 u - \partial_x^2 u + u^3$ is not a linear operator. By direct computation, we check

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v + 3u^2 v + 3vu^2 \Rightarrow \mathcal{L}(u+v) \neq \mathcal{L}u + \mathcal{L}v.$$

(iii) Consider the PDE $\partial_t^2 u - \partial_x^2 u + \partial_x u + u = 0$ where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The PDE is linear equation since the operator $\mathcal{L}u = \partial_t^2 u - \partial_x^2 u + \partial_x u + u$ is a linear operator. By direct computation, we check

$$\mathcal{L}(u+v) = \left(\partial_t^2 u - \partial_x^2 u + \partial_x u + u\right) + \left(\partial_t^2 v - \partial_x^2 v + \partial_x v + v\right) = \mathcal{L}u + \mathcal{L}v.$$

It is worth mentioning here that the advantages of linearity for the equation $\mathcal{L}u = 0$ are twofold:

(i) If $\{u_j\}_{j=1}^N$ are all solutions, then any linear combination

$$\sum_{j=1}^N c_j u_j(x) : \Omega \to \mathbb{R},$$

is also a solution of the equation.

(ii) If u is a homogeneous solution and v is an inhomogeneous solution, then $u+v$ is also an inhomogeneous solution.

Example 1.5. (i) Consider the PDE $\partial_x^2 u(x,y) = 0$ where $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$. We integrate once to get

$$\partial_x u(x_1, y) = \partial_x u(x_2, y) \quad \text{for all } x_1, x_2 \in \mathbb{R} \Rightarrow \partial_x u(x,y) = f(y) \quad \text{on } \mathbb{R}^2.$$

We integrate again to get

$$u(x,y) = f(y)x + g(y) \quad \text{where } g(y) = u(0,y).$$

This is the solution formula. Note that there are two arbitrary functions ($f(y)$ and $g(y)$) in the formula.

(ii) Consider the PDE $\partial_x^2 u(x,y) + u(x,y) = 0$ where $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$. Fix $y \in \mathbb{R}$, it is really an second ODE with variable x. Solving the ODE, we find

$$u(x,y) = f(y)\sin x + g(y)\cos x \quad \text{on } \mathbb{R}^2.$$

(iii) Consider the PDE $\partial_{xy} u(x,y) = 0$ where $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$. First, we integrate in x regarding y as fixed,

$$\partial_y y(x_1, y) = \partial_y u(x_2, y) \text{ for all } x_1, x_2 \in \mathbb{R} \Rightarrow \partial_y u(x,y) = f(y).$$

Then we integrate in y to get

$$u(x,y) = u(x,0) + \int_0^y f(s)\mathrm{d}s = G(x) + F(y) \quad \text{on } \mathbb{R}^2.$$

1.2. **Initial and boundary conditions.** In general, PDEs have lots of solutions, as we saw in Example 1.5. Recall that, even ordinary differential equations (without any imposing condition) have infinitely many solutions. The solutions to dynamical ordinary differential equations are singled out by the imposition of initial conditions, resulting in an initial value problem. On the other hand, equations modeling equilibrium phenomena require boundary conditions to specify their solutions uniquely, resulting in a boundary value problem. For PDE, we consider the two imposing conditions that similar as ODE case.

Example 1.6. (i) Consider the wave equation with an initial condition at $t = 0$:

$$\begin{cases} \partial_t^2 u(t,x) = \Delta u(t,x), & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x), \end{cases}$$

where $u_0(x)$ is the initial position and $u_1(x)$ is the initial velocity.

(ii) Consider the heat equation with an initial condition at $t = 0$:

$$\partial_t u(t,x) = \Delta u(t,x) \quad \text{with } u(0,x) = u_0(x)$$

where $u_0(x)$ is given function on $\mathbb{R}^n$.

We mention here that the three most important kinds of boundary conditions are:

(i) Dirichlet condition: the solution u is specified on boundary $\partial\Omega$.
(ii) Robin condition: the function $\frac{\partial u}{\partial n} + au$ is specified on boundary $\partial\Omega$.
(iii) Neumann condition: the normal derivative $\frac{\partial u}{\partial n}$ is specified on boundary $\partial\Omega$.

Example 1.7. (i) Consider the Dirichlet problem in a bounded domain:

$$\begin{aligned} \Delta u(x) &= f(x) \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u(x) &= g(x) \quad \text{on } \partial\Omega \subset \mathbb{R}^n, \end{aligned}$$

for $f \in C(\Omega)$ and $g \in C(\partial\Omega)$.
(ii) Consider the Neumann condition problem in a bounded domain:

$$\begin{cases} \Delta u(x) = f(x) & \text{in } \Omega \subset \mathbb{R}^n, \\ \dfrac{\partial u}{\partial n}(x) = g(x) & \text{on } \partial\Omega \subset \mathbb{R}^n. \end{cases}$$

(iii) Consider the mixed initial-boundary problem:

$$\begin{cases} \partial_t u(t,x) = \partial_x^2(t,x) & \text{for } (t,x) \in [0,\infty) \times [0,L], \\ u(0,x) = \phi(x) & \text{for } x \in [0,L], \\ u(t,0) = g(t) & \text{for } t \in [0,\infty), \\ u(t,L) = h(t) & \text{for } t \in [0,\infty), \end{cases}$$

where $\phi(x)$, $h(t)$ and $g(t)$ are given functions.

1.3. **Well-posed problems.** The mathematical term well-posed problem stems from a definition given by 20th-century French mathematician Jacques Hadamard. He stated that mathematical models of physical phenomena should have the following three properties that:

(i) Existence: The problem in fact has a solution;
(ii) Uniqueness: There is at most one solution;
(iii) Stability: Solution depends continuously on the data given in the problem.

Note that, we should carefully define what is a solution to PDE. Indeed, there are at least three types of solution: classical solution, weak solution, strong solution. In this lecture, we focus on the classical solutions for PDE.

Definition 1.8. A function $u : \Omega \to \mathbb{R}$ is called a classical solution to a k-th order PDE if it satisfy this equation at every point of its definition and belong to the function set $\mathcal{C}^k$.

Example 1.9. (i) Consider the wave equation in the cylinder Ω:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(t,x) \quad \text{for } (t,x) \in \Omega, \\ (u, \partial_t u)_{|t=0} = (\psi(x), \phi(x)), \\ u(t,0) = g(t), \quad u(t,L) = h(t) \quad \text{for } t \in [0,\infty), \end{cases}$$

where $\Omega = \{(t,x) \in \mathbb{R}^2 : 0 \le t < \infty \text{ and } 0 \le x \le L\}$. The data for this problem consist of five functions $f(t,x)$, $\psi(x)$, $\phi(x)$, $g(t)$ and $h(t)$. Existence and uniqueness would mean that there is exactly one solution $u(t,x)$ for arbitrary (differentiable) functions $f(t,x)$, $\psi(x)$, $\phi(x)$, $g(t)$ and $h(t)$. Stability would mean that if any of these five functions are perturbed, then u is also changed only slightly.

(ii) Consider the Laplace equation on half plane with boundary condition

$$\begin{cases} \Delta u_n(x,y) = 0 & \text{in } \Omega \\ u_n(x,0) = 0, \quad \dfrac{\partial u_n}{\partial y}(x,0) = e^{-\sqrt{n}} \sin nx, \end{cases}$$

where $\Omega = \big\{(x,y) \in \mathbb{R}^2 : -\infty < x < \infty \text{ and } 0 < y < \infty\big\}$. For any $n \in \mathbb{N}$, we find that the solution is

$$u_n(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny \quad \text{on } \Omega.$$

Note that $\frac{\partial u_n}{\partial y}(x,0) \to 0$ as $n \to \infty$. However for $y \geq 1$, the solutions $u_n(x,y)$ do not tend to 0 as $n \to \infty$ which means that the stability is not true.

1.4. **Types of second-order equations.** Consider the second-order PDE

$$a_{11}\partial_x^2 u + 2a_{12}\partial_{xy} u + a_{22}\partial_y^2 u + a_1\partial_x u + a_2\partial_y u + a_0 u = 0, \tag{1.2}$$

where $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$.

Theorem 1.10. *By a linear transformation of the independent variables, the equation can be reduced to one of three forms, as follows.*

(i) Elliptic case*: If $a_{12}^2 < a_{11}a_{22}$, then the equation (1.2) is reducible to*

$$\partial_x^2 u + \partial_y^2 u + c_1\partial_x u + c_2\partial_y u + c_0 u = 0,$$

where $c_0, c_1, c_2 \in \mathbb{R}$.

(ii) Parabolic case: *If $a_{12}^2 = a_{11}a_{22}$, then the equation (1.2) is reducible to*

$$\partial_x^2 u + c_1\partial_x u + c_2\partial_y u + c_0 u = 0,$$

where $c_0, c_1, c_2 \in \mathbb{R}$.

(iii) Hyperbolic case: *If $a_{12}^2 > a_{11}a_{22}$, then the equation (1.2) is reducible to*

$$\partial_x^2 u - \partial_y^2 u + c_1\partial_x u + c_2\partial_y u + c_0 u = 0,$$

where $c_0, c_1, c_2 \in \mathbb{R}$.

Proof. Case 1: $a_{11}^2 + a_{22}^2 = 0$. In this case, without loss of generality, we assume $2a_{12} = 1$. Therefore, the equation (1.2) is

$$\partial_{xy} u + a_1\partial_x u + a_2\partial_y u + a_0 u = 0, \tag{1.3}$$

where $a_1, a_2, a_0 \in \mathbb{R}$. Consider the linear change of variable

$$\begin{cases} x = \xi + \eta \\ y = \xi - \eta \end{cases} \quad \text{and} \quad \tilde{u}(\xi,\eta) = u(x,y) = u(\xi+\eta, \xi-\eta).$$

By direct computation, we have

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$
$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$

which implies

$$\partial_\xi^2 = \partial_x^2 + \partial_y^2 + 2\partial_{xy}, \quad \partial_\eta^2 = \partial_x^2 + \partial_y^2 - 2\partial_{xy}, \quad \frac{\partial_\xi + \partial_\eta}{2} = \partial_x, \quad \frac{\partial_\xi - \partial_\eta}{2} = \partial_y.$$

Based on the above identities and (1.3), we have

$$\begin{aligned} \partial_\xi^2 \tilde{u}(\xi,\eta) - \partial_\eta^2 \tilde{u}(\xi,\eta) &= 4\left(\partial_{xy} u\right)(\xi+\eta, \xi-\eta) \\ &= -2(a_1+a_2)\partial_\xi \tilde{u}(\xi,\eta) - 2(a_1-a_2)\partial_\eta \tilde{u}(\xi,\eta) + 4a_0 \tilde{u}(\xi,\eta), \end{aligned}$$

which is Hyperbolic type equation.

Case 2: $a_{11}^2 + a_{22}^2 \neq 0$. In this case, without loss of generality, we assume $a_{11} = 1$. We rewrite (1.2) as

$$(\partial_x + a_{12}\partial_y)^2 u + (a_{22} - a_{12}^2)\partial_y^2 u + a_1(\partial_x + a_{12}\partial_y)u + (a_2 - a_1 a_{12})\partial_y u + a_0 u = 0.$$

When $b = \left|a_{22} - a_{12}^2\right|^{\frac{1}{2}} > 0$. Consider the linear change of variable

$$\begin{cases} x = \xi \\ y = a_{12}\xi + b\eta \end{cases} \quad \text{and} \quad \tilde{u}(\xi, \eta) = u(x, y) = u(\xi, a_{12}\xi + b\eta).$$

By direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial \eta} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} = b\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}. \end{aligned}$$

which implies

$$\partial_\xi^2 \tilde{u} + \operatorname{sign}(a_{22} - a_{12}^2)\partial_\eta^2 \tilde{u} + a_1 \partial_\xi \tilde{u} + (a_2 - a_1 a_{12})|a_{22} - a_{12}^2|^{-\frac{1}{2}}\partial_\eta \tilde{u} + a_0 \tilde{u} = 0.$$

Therefore, the type of equation (1.2) dependent on the sign of $(a_{22} - a_{12}^2)$.

When $b = \left|a_{22} - a_{12}^2\right|^{\frac{1}{2}} = 0$ and $a_{12} \neq 0$. Consider the linear change of variable

$$\begin{cases} x = \xi \\ y = a_{12}\xi + \eta \end{cases} \quad \text{and} \quad \tilde{u}(\xi, \eta) = u(x, y) = u(\xi, a_{12}\xi + \eta).$$

By direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial \eta} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}. \end{aligned}$$

which implies

$$\partial_\xi^2 \tilde{u} + a_2 \partial_\xi \tilde{u} + (a_2 - a_1 a_{12})\partial_\eta \tilde{u} + a_0 \tilde{u} = 0.$$

Therefore, the equation (1.2) is Parabolic-type equation.

When $b = \left|a_{22} - a_{12}^2\right|^{\frac{1}{2}} = 0$ and $a_{12} = 0$, we find that $a_{22} = 0$ which implies the equation (1.2) is Parabolic type equation. □

The same argument can be done in any number of variables, using linear algebra. Consider the second-order PDE

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + cu = 0. \tag{1.4}$$

where a_{ij}, b_i, $c \in \mathbb{R}$. Since the mixed derivatives are equal, we may as well assume that $a_{ij} = a_{ji}$ for any $i, j = 1, \cdots, n$. Denote $A = (a_{ij})_{i,j=1,\cdots,n}$ be a $n \times n$ symmetric matrix. Using a theorem from linear algebra, we know that there exists a matrix B with $\det B = 1$ such that

$$BAB^T = \operatorname{diag}(d_1, d_2, \ldots, d_n).$$

Recall that, the real numbers $d_1, d_2, \ldots, d_n$ are the eigenvalues of A.

Definition 1.11. The second-order PDE (1.4) is called elliptic if all the eigenvalues $d_1, \ldots, d_n$ are positive or are all negative. The second-order PDE (1.4) is called hyperbolic if none of the $d_1, \ldots, d_n$ vanish and one of them has the opposite sign from the $(n-1)$ others. If none vanish, but at least two of them are positive and at least two are negative, it is called ultrahyperbolic. If exactly one of the eigenvalues is zero and all the others have the same sign, the PDE is called parabolic.

2. Four important linear PDE

In this section, we will introduce four fundamental linear PDE. These are

$$\begin{aligned} &\text{the transport equation} && \partial_t u + b\cdot\nabla_x u = f,\\ &\text{the Laplace's equation} && -\Delta_x u = f,\\ &\text{the 1D heat equation} && \partial_t u - \partial_x^2 u = f,\\ &\text{the 1D wave equation} && \partial_t^2 u - \partial_x^2 u = f. \end{aligned} \tag{2.1}$$

In this section, the presentation is usually close to [1, Chapter 2] and [4, Chapter 2, Chapter 3, Chapter 6 and Chapter 7].

2.1. **Transport equation.** Let $n \in \{1,2,3\}$. Consider the following PDE

$$\partial_t u + b\cdot\nabla u = 0 \quad \text{in } (t,x)\in(0,\infty)\times\mathbb{R}^n, \tag{2.2}$$

where $b = (b_1,\dots,b_n)\in\mathbb{R}^n$ and $u : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is unknown.
We use the so called method of characteristics to find solution of (2.2). More precisely, we consider a flow of u by fixing any point $(t,x)\in(0,\infty)\times\mathbb{R}^n$ and defining

$$z(s) = u(t+s, x+sb), \quad \text{for all } s \ge -t.$$

By direct computation and using the equation (2.2),

$$\frac{\mathrm{d}}{\mathrm{d}s} z(s) = (\partial_t u)(t+s,x+sb) + (b\cdot\nabla u)(t+s,x+sb) = 0.$$

Therefore, $z(\cdot)$ is a constant function of s, and consequently for each point (t,x), u is constant on the line through (t,x) with the direction $(1,b)\in\mathbb{R}^{1+n}$. Hence if we know the value of u at any point on each such line, we know its value everywhere in $(0,\infty)\times\mathbb{R}^n$.

2.1.1. **Initial-value problem**. Consider the initial-value problem

$$\begin{cases} \partial_t u + b\cdot\nabla u = 0 & \text{for } (t,x)\in(0,\infty)\times\mathbb{R}^n,\\ u(0,x) = g(x) & \text{for all } x\in\mathbb{R}^n. \end{cases} \tag{2.3}$$

Here $b\in\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}$ are known, and the problem is to compute u. Fix $(t,x)\in(0,\infty)\times\mathbb{R}^n$, the line through (t,x) with direction $(1,b)$ is represented by $(t+s, x+bs)$ for $s\in\mathbb{R}$. This line hits the plane $\Lambda = \{t=0\}\times\mathbb{R}^n$ when $s=-t$, at the point $(0, x-tb)$. Since u is constant on the line and $u(0,x-tb) = g(x-tb)$, we deduce

$$u(t,x) = g(x-tb) \quad \text{for all } (t,x)\in[0,\infty)\times\mathbb{R}^n. \tag{2.4}$$

So, if (2.3) has a sufficiently regular solution u, it must certainly be given by (2.4). And conversely, it is easy to check directly that if g is C^1, then u defined by (2.4) is indeed a solution of (2.3).

Example 2.1. (i) Consider the PDE

$$\begin{cases} \partial_t u - \partial_x u = 0 & \text{for } (t,x)\in(0,\infty)\times\mathbb{R},\\ u(0,x) = x^3 & \text{for } x\in\mathbb{R}. \end{cases}$$

From (2.4), we have

$$u(t,x) = (x+t)^3 \quad \text{for } (t,x)\in(0,\infty)\times\mathbb{R}.$$

We check this solution by direct computation,

$$\partial_t u(t,x) = \partial_x u(t,x) = 3(x+t)^2 \Rightarrow \partial_t u(t,x) - \partial_x u(t,x) = 0.$$

(ii) Consider the PDE

$$\begin{cases} \partial_t u + x\partial_x u = 0 & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = x^3 & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.5}$$

Consider the flow $s \mapsto (t(s), x(s))$ for unknown $t(s)$ and $x(s)$. Denote

$$z(s) = u(t(s), x(s)) \quad \text{for } s \in \mathbb{R}.$$

By direct computation,

$$\frac{\mathrm{d}}{\mathrm{d}s}z(s) = \frac{\mathrm{d}t}{\mathrm{d}s}(\partial_t u)(t(s),x(s)) + \frac{\mathrm{d}x}{\mathrm{d}s}(\partial_x u)(t(s),x(s)).$$

From (2.5), we know that the unknown function u is invariant on the flow $(t(s), x(s))$ which satisfy

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1 \ \text{ and } \ \frac{\mathrm{d}x}{\mathrm{d}s} = x \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = x \Rightarrow x = \pm e^{t+c}.$$

Fix $(t_0, x_0) \in [0,\infty)\times\mathbb{R}$, we know that $e^c = \pm x_0 e^{-t_0}$ which implies

$$\text{for } t = 0, \text{ we have } \ x = x_0 e^{-t_0} \Rightarrow u(t,x) = x^3 e^{-3t} \text{ on } (t,x) \in (0,\infty)\times\mathbb{R}.$$

2.1.2. **Nonhomogeneous Problem**. Consider the nonhomogeneous problem

$$\begin{cases} \partial_t u + b\cdot\nabla u = f & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = g(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{2.6}$$

As before fix $(t,x) \in (0,\infty)\times\mathbb{R}^n$ and set $z(s) = u(t+s, x+sb)$. Using (2.6),

$$\frac{\mathrm{d}}{\mathrm{d}s}z(s) = (\partial_t u)(t+s,x+sb) + (b\cdot\nabla u)(t+s,x+sb) = f(t+s,x+sb).$$

Integrating above identity on $[-t,0]$ for all $t \in (0,\infty)$, we have

$$u(t,x) - g(x-tb) = \int_{-t}^0 f(t+s,x+sb)\mathrm{d}s = \int_0^t f(s,x+(s-t)b)\mathrm{d}s,$$

which implies

$$u(t,x) = g(x-tb) + \int_0^t f(s,x+(s-t)b)\mathrm{d}s \quad \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^n.$$

2.2. **Laplace's equation.** Let $n = 2,3$. Consider the Laplace's equation

$$\Delta u = \sum_{i=1}^n \partial_{x_i}^2 u = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \tag{2.7}$$

and the Poisson's equation

$$-\Delta u = -\sum_{i=1}^n \partial_{x_i}^2 u = f \quad \text{for } x \in \Omega \subset \mathbb{R}^n. \tag{2.8}$$

Definition 2.2. A $C^2(\mathbb{R}^n)$ function u satisfying (2.7) is called a harmonic function.

2.2.1. **Mean-value formulas**. Consider an open set $\Omega \subset \mathbb{R}^n$ and suppose u is a real function within Ω.

Definition 2.3. For $u \in C(\Omega)$, we define

(i) u satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)\mathrm{d}S_y \quad \text{for any } B_r(x) \subset \Omega; \tag{2.9}$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n r^n}\int_{B_r(x)} u(y)\mathrm{d}y \quad \text{for any } B_r(x) \subset \Omega. \tag{2.10}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$ ($\omega_2 = 2\pi$ and $\omega_3 = 4\pi$).

Lemma 2.4. *These two definitions* (i) *and* (ii) *are equivalent.*

Proof. First, from (2.9), we find

$$u(x)r^{n-1} = \frac{1}{\omega_n}\int_{\partial B_r(x)} u(y)\mathrm{d}S_y \quad \text{for any } B_r(x) \subset \Omega.$$

Integrating the above identity on $[0, r]$, we have

$$u(x)\frac{r^n}{n} = \frac{1}{\omega_n}\int_0^r \int_{\partial B_{\bar{r}}(x)} u(y)\mathrm{d}S_y \mathrm{d}\bar{r} = \frac{1}{\omega_n}\int_{B_r(x)} u(y)\mathrm{d}y,$$

which means that (i) implies (ii).
Second, from (2.10), we have

$$u(x)r^n = \frac{n}{\omega_n}\int_{B_r(x)} u(y)\mathrm{d}y = \frac{n}{\omega_n}\int_0^r \int_{\partial B_{\bar{r}}(x)} u(y)\mathrm{d}S_y \mathrm{d}\bar{r}.$$

We may differentiate the above identity to get

$$nu(x)r^{n-1} = \frac{n}{\omega_n}\int_{\partial B_r(x)} u(y)\mathrm{d}S_y \quad \text{for any } B_r(x) \subset \Omega,$$

which means that (ii) implies (i). □

Remark 2.5. By change of variable, we may write the mean-value properties in the following equivalent ways:

(i) u satisfies the first mean-value property if

$$u(x) = \frac{1}{\omega_n}\int_{\partial B_1(0)} u(x + r\omega)\mathrm{d}S_\omega \quad \text{for any } B_r(x) \subset \Omega.$$

(ii) u satisfies the second mean-value property if

$$u(x) = \frac{n}{\omega_n}\int_{B_1(0)} u(x + ry)\mathrm{d}y \quad \text{for any } B_r(x) \subset \Omega.$$

Lemma 2.6. *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then u satisfies the mean-value property in Ω.*

Proof. Consider

$$\phi(r) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)\mathrm{d}S_y = \frac{1}{\omega_n}\int_{\partial B_1(0)} u(x + r\omega)\mathrm{d}S_\omega.$$

Note that

$$u(x) = \lim_{\rho\to 0}\left(\frac{1}{\omega_n \rho^{n-1}}\int_{\partial B_\rho(x)} u(y)\mathrm{d}S_y\right) = \lim_{\rho\to 0}\phi(\rho).$$

Using the change of variable and Green's formula, we compute

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}r} &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \omega\cdot\nabla u(x + r\omega)\mathrm{d}S_\omega \\
&= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} \nabla u(y)\cdot\frac{y - x}{r}\mathrm{d}S_y \\
&= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} \frac{\partial u}{\partial n}\mathrm{d}S_y = \frac{1}{\omega_n r^{n-1}}\int_{B_r(x)} \Delta u(y)\mathrm{d}y = 0.
\end{aligned}$$

Thus $\phi : \mathbb{R} \to \mathbb{R}$ is constant, and so

$$u(x) = \lim_{\rho\to 0}\phi(\rho) = \phi(r) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)\mathrm{d}S_y.$$

□

For a function u satisfying the mean-value property, u is not required to be smooth. However a harmonic function is required to be C^2. We prove these two are equivalent.

Lemma 2.7. *If $u \in C(\Omega)$ has mean-value property in Ω, then u is smooth and harmonic in Ω.*

Proof. Choose $\varphi \in C_0^\infty(B_1(0))$ with

$$\int_{B_1(0)} \varphi(x)\mathrm{d}x = 1 \quad \text{and} \quad \varphi(x) = \psi(|x|).$$

Note that

$$\omega_n \int_0^1 r^{n-1}\psi(r)\mathrm{d}r = \int_0^1 \int_{\partial B_r(0)} \varphi(x)\mathrm{d}S_x\mathrm{d}r = \int_{B_1(0)} \varphi(x)\mathrm{d}x = 1. \tag{2.11}$$

Denote $\varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon^n}\varphi(\frac{\cdot}{\varepsilon})$ for $\varepsilon > 0$. For any $x \in \Omega$, we consider $0 < \varepsilon < \mathrm{dist}(x, \partial\Omega) < +\infty$. By the change of variable, we have

$$\begin{aligned}\int_\Omega u(y)\varphi_\varepsilon(y-x)\mathrm{d}y &= \int_{B_\varepsilon(0)} u(x+y)\varphi_\varepsilon(y)\mathrm{d}y \\ &= \frac{1}{\varepsilon^n}\int_{B_\varepsilon(0)} u(x+y)\varphi\left(\frac{y}{\varepsilon}\right)\mathrm{d}y = \int_{B_1(0)} u(x+\varepsilon y)\varphi(y)\mathrm{d}y.\end{aligned}$$

Therefore, from (2.11) and the mean-value property of u, we have

$$\begin{aligned}\int_\Omega u(y)\varphi_\varepsilon(y-x)\mathrm{d}y &= \int_{B_1(0)} u(x+\varepsilon y)\varphi(y)\mathrm{d}y \\ &= \int_0^1 r^{n-1}\int_{\partial B_1(0)} u(x+\varepsilon r\omega)\varphi(r\omega)\mathrm{d}S_\omega\mathrm{d}r \\ &= \int_0^1 \psi(r)r^{n-1}\int_{\partial B_1(0)} u(x+\varepsilon r\omega)\mathrm{d}S_\omega\mathrm{d}r = u(x).\end{aligned}$$

Hence we have

$$u(x) = (u * \varphi_\varepsilon)(x) \quad \text{for any } x \in \Omega_\varepsilon = \{y \in \Omega; \mathrm{dist}(y, \partial\Omega) > \varepsilon\},$$

which implies $u \in C^\infty$. Moreover, using again the mean-value property of u and the Green's formula,

$$\begin{aligned}\int_{B_r(x)} \Delta u(y)\mathrm{d}y &= r^{n-1}\frac{\partial}{\partial r}\int_{|\omega|=1} u(x+r\omega)\mathrm{d}S_\omega \\ &= r^{n-1}\frac{\partial}{\partial r}(\omega_n u(x)) = 0 \quad \text{for any } B_r(x) \subset \Omega,\end{aligned}$$

which implies $\Delta u = 0$ in Ω. □

Now we prove the maximum principle for the functions satisfying mean-value properties.

Proposition 2.8. *If $u \in C\left(\bar{\Omega}\right)$ satisfies the mean-value property in Ω, then u assumes its maximum and minimum only on $\partial\Omega$ unless u is constant.*

Proof. We only prove for the maximum since the minimum case is similar. Let

$$\Sigma = \left\{x \in \Omega : u(x) = M = \max_{x\in\bar{\Omega}} u\right\} \subset \Omega.$$

First, from the definition of Σ and $u \in C(\bar{\Omega})$, we know that the set Σ is relatively closed. Second, for any $x_0 \in \Sigma$, we choose r small enough such that $B_r(x_0) \subset \Omega$. Using the mean-value property of u, we have

$$M = u(x_0) = \frac{n}{\omega_n r^n} \int_{B_r(x_0)} u(y)\mathrm{d}y \leq \frac{Mn}{\omega_n r^n} \int_{B_r(x_0)} 1\mathrm{d}y = M,$$

which implies $u(x) = M$ for all $x \in B_r(x_0)$. Therefore, Σ is both open and closed in Ω which implies either $\Sigma = \emptyset$ or $\Sigma = \Omega$. $\square$

An important application of the maximum principle is establishing the uniqueness of solutions to the Dirichlet problem in a bounded domain.

Theorem 2.9. *Let $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the Dirichlet problem*

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ \quad u(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Proof. If u_1 and u_2 both satisfy (2.12), apply Proposition 2.8 to the function $v = u_1 - u_2$. $\square$

Remark 2.10. In general, the uniqueness does not hold for an unbounded domain. For example, we consider the following Dirichlet problem in the unbounded domain Ω,

$$\begin{cases} -\Delta u(x) = 0 & \text{in } \Omega, \\ \quad u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^n; |x| > 1\}$. It is obviously that the function $u(x) \equiv 0$ is a solution. For $n = 2$, by direct computation, $u(x) = \log|x|$ is also a solution. For $n = 3$, by direct computation, $u(x) = |x|^{-1} - 1$ is also a solution.

We finish this subsection by Harnack inequality.

Theorem 2.11. *Suppose u is harmonic in Ω. Then for any compact subset K of Ω there exists a positive constant $C = C(\Omega, K)$ such that if $u \geq 0$ in Ω, then*

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y) \quad \text{for any } x, y \in K.$$

Proof. Let $r = \frac{1}{4}\mathrm{dist}(K, \partial\Omega)$. Choose $x, y \in K$ with $|x - y| \leq r$. Then, from the mean-value property, we have

$$u(x) = \frac{n}{\omega_n (2r)^n} \int_{B_{2r}(x)} u(z)\mathrm{d}z \geq \frac{n}{2^n \omega_n r^n} \int_{B_r(y)} u(z)\mathrm{d}z = \frac{1}{2^n} u(y).$$

Thus, we have $2^{-n}u(y) \leq u(x) \leq 2^n u(y)$ for all $x, y \in K$ with $|x - y| \leq r$. Since K is connected and compact, we can cover K by a chain of finitely many balls $\{B_i\}_1^N$, each of which has radius $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, \cdots, N$. Then $2^{-nN}u(y) \leq u(x) \leq 2^{nN}u(y)$ for all $x, y \in K$. $\square$

2.2.2. **Green function**. In this subsection, we introduce the Green function which is a tool to solve Poisson's equation. Consider a harmonic function u in $\mathbb{R}^n$ (with $n = 2, 3$) which depends only on $r = |x - a|$ for some fixed $a \in \mathbb{R}^n$. We set $v(r) = u(x)$. This implies

$$\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \Rightarrow r\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + (n-1)\frac{\mathrm{d}v}{\mathrm{d}r} = 0,$$

and hence

$$v(r) = \begin{cases} c_1 + c_2 \log r, & n = 2, \\ c_3 + c_4 r^{-1}, & n = 3, \end{cases}$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$. We are interested in a function with a singularity such that

$$\int_{\partial B_r(a)} \frac{\partial v}{\partial r} \mathrm{d}S = 1 \quad \text{for any } r > 0.$$

Hence we set for any fixed $a \in \mathbb{R}^n$

$$\Gamma(a,x) = \frac{1}{2\pi} \log |x-a|, \qquad \text{for } n = 2,$$
$$\Gamma(a,x) = -\frac{1}{4\pi} |x-a|^{-1}, \qquad \text{for } n = 3.$$

To summarize, we have that for fixed $a \in \mathbb{R}^n$, $\Gamma(a,x)$ is harmonic at $x \neq a$, that is,

$$\Delta_x \Gamma(a,x) = 0 \quad \text{for any } x \neq a,$$

and has a singularity at $x = a$. Moreover, it satisfies

$$\int_{\partial B_r(a)} \frac{\partial \Gamma}{\partial n_x}(a,x) \mathrm{d}S_x = 1 \quad \text{for any } r > 0.$$

By direct computation, we have the following Green's identity.

Lemma 2.12. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ and that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then for any $a \in \Omega$ there holds*

$$u(a) = \int_\Omega \Gamma(a,x) \Delta u(x) \mathrm{d}x - \int_{\partial\Omega} \left(\Gamma(a,x) \frac{\partial u}{\partial n_x}(x) - u(x) \frac{\partial \Gamma}{\partial n_x}(a,x) \right) \mathrm{d}S_x.$$

Proof. We apply Green's formula to u and $\Gamma(a,\cdot)$ in the domain $\Omega \setminus B_r(a)$ for small $r > 0$ and get

$$\begin{aligned} &\int_{\Omega \setminus B_r(a)} (\Gamma \Delta u - u \Delta \Gamma) \mathrm{d}x \\ &= \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \right) \mathrm{d}S_x - \int_{\partial B_r(a)} \left(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \right) \mathrm{d}S_x. \end{aligned}$$

Note that $\Delta \Gamma = 0$ in $\Omega \setminus B_r(a)$. Then we have

$$\lim_{r \to 0} \int_{\partial B_r(a)} \left(u \frac{\partial \Gamma}{\partial n} - \Gamma \frac{\partial u}{\partial n} \right) \mathrm{d}S_x = \int_\Omega \Gamma \Delta u \mathrm{d}x - \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \right) \mathrm{d}S_x.$$

For $n = 3$, we get by definition of Γ,

$$\begin{aligned} \left| \int_{\partial B_r(a)} \Gamma \frac{\partial u}{\partial n} \mathrm{d}S_x \right| &= \left| \frac{1}{4\pi r} \int_{\partial B_r(a)} \frac{\partial u}{\partial n} \mathrm{d}S_x \right| \\ &\leq r \max_{x \in \partial B_r(a)} |\nabla u| \to 0 \quad \text{as } r \to 0, \end{aligned}$$

$$\int_{\partial B_r(a)} u \frac{\partial \Gamma}{\partial n_x} \mathrm{d}S_x = \frac{1}{4\pi r^2} \int_{\partial B_r(a)} u(x) \mathrm{d}S_x \to u(a) \quad \text{as } r \to 0.$$

We get the same conclusion for $n = 2$ in the same way. □

Remark 2.13. (i) For $a \notin \bar{\Omega}$, the expression in the right side gives 0.
(ii) For any $a \in \Omega$, $\Gamma(a,\cdot)$ is integrable in Ω although it has a singularity.
(iii) By letting $u = 1$, we have $\int_{\partial\Omega} \frac{\partial \Gamma}{\partial n_x}(a,x) \mathrm{d}S_x = 1$ for any $a \in \Omega$.

Now we begin to introduce the Green's function. Suppose Ω is bounded domain in $\mathbb{R}^n$. Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. From Lemma 2.12, we know that, for all $x \in \Omega$,

$$u(x) = \int_\Omega \Gamma(x,y) \Delta u(y) \mathrm{d}y - \int_{\partial\Omega} \left(\Gamma(x,y) \frac{\partial u}{\partial n_y}(y) - u(y) \frac{\partial \Gamma}{\partial n_y}(x,y) \right) \mathrm{d}S_y. \quad (2.13)$$

If u solves the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$, then u can be expressed in terms of f and φ, with one unknown term. We want to eliminate this term by adjusting Γ.
For any $x \in \Omega$, we consider

$$G(x,y) = \Gamma(x,y) + \Psi(x,y) \quad \text{for } (x,y) \in \Omega \times \bar{\Omega},$$

for some $\Psi(x,\cdot) \in C^2(\bar{\Omega})$ with $\Delta_y \Psi(x,y) = 0$ in Ω. Using the Green's formula for $\Psi(x,y)$, we have

$$\int_\Omega \Psi(x,y)\Delta u(y)\mathrm{d}y = \int_{\partial\Omega} \left(\Psi(x,y)\frac{\partial u}{\partial n_y} - u(y)\frac{\partial \Psi}{\partial n_y}(x,y) \right) \mathrm{d}S_y. \tag{2.15}$$

Combining (2.13) and (2.15), we know that

$$u(x) = \int_\Omega G(x,y)\Delta u(y)\mathrm{d}y - \int_{\partial\Omega} \left(G(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial G}{\partial n_y}(x,y) \right) \mathrm{d}S_y.$$

Now by choosing Ψ appropriately, we are led to the important concept of Green's function. For each fixed $x \in \Omega$ choose $\Psi(x,\cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} \Delta_y \Psi(x,y) = 0 & \text{for } y \in \Omega, \\ \Psi(x,y) = -\Gamma(x,y) & \text{for } y \in \partial\Omega. \end{cases} \tag{2.16}$$

If such a Ψ exists, then the solution u to the Dirichlet problem (4.14) can be expressed as

$$u(x) = \int_\Omega G(x,y)f(y)\mathrm{d}y + \int_{\partial\Omega} \varphi(y)\frac{\partial G}{\partial n_y}(x,y)\mathrm{d}S_y. \tag{2.17}$$

Now we discuss some properties of G as a function of x and y. Our first observation is that the Green's function is unique. This is proved by the maximum principle since the difference of two Green's functions are harmonic in Ω with zero boundary value. Moreover, the function G is symmetric in the variables x and y:

Proposition 2.14. *Green's function $G(x,y)$ is symmetric in $\Omega\times\Omega$; that is $G(x,y) = G(y,x)$ for any $x \neq y \in \Omega$.*

Proof. Pick $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$. Choose $r > 0$ small such that $B_r(x_1) \cap B_r(x_2) = \emptyset$. Denote $G_1(y) = G(x_1, y)$ and $G_2(y) = G(x_2, y)$. We apply Green's formula in $\Omega \setminus B_r(x_1) \cup B_r(x_2)$ and get

$$\begin{aligned} &\int_{\partial B_r(x_1)} \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n} \right) \mathrm{d}S + \int_{\partial B_r(x_2)} \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n} \right) \mathrm{d}S \\ &= \int_{\partial\Omega} \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n} \right) \mathrm{d}S - \int_{\Omega\setminus B_r(x_1)\cup B_r(x_2)} \left(G_1\Delta G_2 - G_2\Delta G_1 \right). \end{aligned}$$

Since G_i is harmonic for $y \neq x_i$, $i = 1, 2$, and vanishes on $\partial\Omega$, we have

$$\int_{\partial B_r(x_1)} \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n} \right) \mathrm{d}S + \int_{\partial B_r(x_2)} \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n} \right) \mathrm{d}S = 0.$$

Note that, from the definition of G_1 and G_2, as $r \to 0$,

$$\begin{aligned} \int_{\partial B_r(x_1)} G_1\frac{\partial G_2}{\partial n}\mathrm{d}S &= \int_{\partial B_r(x_1)} \left(\Gamma(x_1,y) + \Psi(x_1,y)\right)\frac{\partial G_2}{\partial n}(y)\mathrm{d}S_y \to 0, \\ \int_{\partial B_r(x_2)} G_2\frac{\partial G_1}{\partial n}\mathrm{d}S &= \int_{\partial B_r(x_2)} \left(\Gamma(x_2,y) + \Psi(x_2,y)\right)\frac{\partial G_1}{\partial n}(y)\mathrm{d}S_y \to 0, \end{aligned}$$

$$\int_{\partial B_r(x_1)} G_2 \frac{\partial G_1}{\partial n} \mathrm{d}S = \int_{\partial B_r(x_1)} G_2(y) \left(\frac{\partial \Gamma}{\partial n}(x_1, y) + \frac{\partial \Psi}{\partial n}(x_1, y) \right) \mathrm{d}S_y \to G_2(x_1),$$
$$\int_{\partial B_r(x_2)} G_1 \frac{\partial G_2}{\partial n} \mathrm{d}S = \int_{\partial B_r(x_2)} G_1(y) \left(\frac{\partial \Gamma}{\partial n}(x_2, y) + \frac{\partial \Psi}{\partial n}(x_2, y) \right) \mathrm{d}S_y \to G_1(x_2).$$

Therefore, we have $G_2(x_1) = G_1(x_2)$ which means $G(x_1, x_2) = G(x_2, x_1)$. □

In the next two subsections, we will build Green's functions for two regions with simple geometry, namely the ball $B_R(0)$ and the half-space $\mathbb{R}^n_+$. Everything depends upon our explicitly solving the corrector problem (2.16) in these regions, and this in turn depends upon some geometric reflection tricks.

A. Green's function for a ball. To construct Green's function for the ball $B_R(0)$, we will employ a kind of reflection through the sphere $\partial B_R(0)$.

Definition 2.15. Given a fixed sphere $\partial B_R(0)$, the inversion of a point x in $\mathbb{R}^n$ is defined to be

$$x^* = \frac{R^2}{|x|^2} x.$$

Remark 2.16. A useful effect of this inversion is that the origin 0 is the image of ∞ , and ∞ is the image of 0. Under this inversion, spheres are transformed into spheres, and the exterior of a sphere is transformed to the interior, and vice versa.

We now employ inversion through the sphere to compute Green's function for the ball $B_R(0)$. Fix $x \in B_R(0)$. Recall that, we must find a corrector function $\Psi(x, y)$ solving

$$\begin{cases} \Delta_y \Psi(x, y) = 0 & \text{for } y \in B_R(0), \\ \Psi(x, y) = -\Gamma(x, y) & \text{for } y \in \partial B_R(0), \end{cases}$$

then the Green's function will be

$$G(x, y) = \Gamma(x, y) + \Psi(x, y) \quad \text{for } (x, y) \in B_R(0) \times \bar{B}_R(0).$$

The idea now is to "invert the singularity" from $x \in B_R(0)$ to $x^* \neq B_R(0)$. Assume for the moment $n = 3$. Now the mapping $y \mapsto \Gamma(x^*, y)$ is harmonic for $y \neq x^*$. Therefore, we have that the mapping $y \mapsto |\frac{x}{R}|^{-1}\Gamma(x^*, y)$ is also harmonic for $y \neq x^*$ which implies

$$\Psi(x, y) = \Gamma\left(\left|\frac{x}{R}\right| x^*, \left|\frac{x}{R}\right| y \right)$$

is harmonic in $B_R(0)$. Moreover, if $y \in \partial B_R(0)$ and $x \neq 0$, we have

$$\begin{aligned} \left|\left|\frac{x}{R}\right| x^* - \left|\frac{x}{R}\right| y\right|^2 &= \frac{|x|^2}{R^2} \left(R^2 - 2\frac{R^2}{|x|^2} x \cdot y + \frac{R^4}{|x|^2} \right) \\ &= \left(|x|^2 - 2x \cdot y + R^2 \right) = |x - y|^2, \end{aligned}$$

which implies $\Psi(x, y) = \Gamma(x, y)$ for $y \in \partial B_R(0)$. This identity is also true for $n = 2$. Based on the above argument, we have the following Proposition.

Proposition 2.17. *The Green's function for the ball $B_R(0)$ is given by*

$$G(x, y) = \frac{1}{2\pi} \left(\log|x - y| - \log\left| \frac{R}{|x|} x - \frac{|x|}{R} y \right| \right), \qquad \textit{for } n = 2,$$
$$G(x, y) = -\frac{1}{4\pi} \left(|x - y|^{-1} - \left| \frac{R}{|x|} x - \frac{|x|}{R} y \right|^{-1} \right), \qquad \textit{for } n = 3.$$

Next we calculate the normal derivative of Green's function on the sphere.

Lemma 2.18. *Suppose G is the Green's function in $B_R(0)$. Then there holds*

$$\frac{\partial G}{\partial n}(x,y)=\frac{R^2-|x|^2}{\omega_n R|x-y|^n},\quad \text{for any } x\in B_R \text{ and } y\in\partial B_R.$$

Proof. We just consider the case $n=3$. By direct computation,

$$\partial_{y_i}G(x,y)=-\frac{1}{4\pi}\frac{x_i-y_i}{|x-y|^3}+\frac{|x|}{4\pi R}\frac{\left(\frac{R}{|x|}x_i-\frac{|x|}{R}y_i\right)}{|x-y|^3}=\frac{y_i}{4\pi R^2}\frac{R^2-|x|^2}{|x-y|^3}.$$

On the other hand, we have $n_i=\frac{y_i}{R}$ for $|y|=R$. Thus

$$\frac{\partial G}{\partial n}(x,y)=\sum_{i=1}^{3}\vec{n}\cdot\nabla_y G(x,y)=\frac{1}{4\pi R}\frac{R^2-|x|^2}{|x-y|^3}.$$

□

We denote by $K(x,y)$ the function in Lemma 2.18 for $x\in\Omega$ and $y\in\partial\Omega$. It is called a Poisson Kernel and has the following properties:

(i) $K(x,y)$ is smooth for $x\neq y$;
(ii) $K(x,y)>0$ for $|x|<R$;
(iii) $\int_{|y|=R}K(x,y)\mathrm{d}S_y=1$ for any $|x|<R$.

The following result gives the existence of harmonic functions in balls with prescribed Dirichlet boundary value.

Theorem 2.19 (Poisson Integral Formula)**.** *For $\varphi\in C\left(\partial B_R(0)\right)$, the function u defined by*

$$u(x)=\begin{cases}\displaystyle\int_{\partial B_R(0)}K(x,y)\varphi(y)\mathrm{d}S_y, & |x|<R,\\ \varphi(x), & |x|=R,\end{cases}$$

satisfies $u\in C(\bar{B}_R(0))\cap C^\infty(B_R(0))$ and

$$\begin{cases}\Delta u=0 & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega.\end{cases}$$

Proof. The proof is based on the properties of $K(x,y)$ and we left it as an exercise. □

Remark 2.20. In the poisson integral formula, by letting $x=0$, we have

$$u(0)=\frac{1}{\omega_n R^{n-1}}\int_{\partial B_R(0)}\varphi(y)\mathrm{d}S_y,$$

which is the mean value property.

Now, we introduce the Harnack's Inequality for harmonic equation in a ball.

Lemma 2.21 (Harnack's Inequality)**.** *Suppose u is harmonic in $B_R(x_0)$ and $u\geq 0$. Then there holds*

$$\left(\frac{R}{R+r}\right)^{n-2}\frac{R-r}{R+r}u(x_0)\leq u(x)\leq\left(\frac{R}{R-r}\right)^{n-2}\frac{R+r}{R-r}u(x_0),$$

where $r=|x-x_0|<R$.

Proof. We may assume $x_0=0$ and $u\in C\left(\bar{B}_R\right)$. Note that u is given by the Poisson integral formula

$$u(x)=\frac{1}{\omega_n R}\int_{\partial B_R}\frac{R^2-|x|^2}{|x-y|^n}u(y)\mathrm{d}S_y.$$

Sine $R-|x|\le|y-x|\le R+|x|$ for $|y|=R$, we have

$$\frac{1}{\omega_n R}\frac{R-|x|}{R+|x|}\left(\frac{1}{R+|x|}\right)^{n-2}\int_{\partial B_R}u(y)\mathrm{d}S_y\le u(x),$$
$$\frac{1}{\omega_n R}\frac{R+|x|}{R-|x|}\left(\frac{1}{R-|x|}\right)^{n-2}\int_{\partial B_R}u(y)\mathrm{d}S_y\ge u(x).$$

Recall the following mean value property

$$u(0)=\frac{1}{\omega_n R^{n-1}}\int_{\partial B_R}u(y)\mathrm{d}S_y.$$

Based on the above inequalities and identity, we finish the proof. □

Corollary 2.22. *If harmonic function u in $\mathbb{R}^n$ is bounded above and below, then $u\equiv C$.*

Proof. We assume $u\ge 0$ in $\mathbb{R}^n$. Taking any point $x\in\mathbb{R}^n$ and applying Lemma 2.21 to any ball $B_R(0)$ with $R>|x|$, we have

$$\left(\frac{R}{R+|x|}\right)^{n-2}\frac{R-|x|}{R+|x|}u(0)\le u(x)\le\left(\frac{R}{R-|x|}\right)^{n-2}\frac{R+|x|}{R-|x|}u(0),$$

which implies $u(x)=u(0)$ by letting $R\to\infty$. □

B. Green's function for a half-space. Consider the half-space

$$\mathbb{R}^n_+=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n|x_n>0\}.$$

Although this region is unbound, and so the calculations in the subsection 2.2.2 do not directly apply, we will attempt nevertheless to build Green's function using the ideas developed earlier. Later of course, we must check directly that the corresponding representation formula is valid.

Definition 2.23. For $x=(x_1,\dots,x_n)\in\mathbb{R}^n_+$, its reflection in the plane $\partial\mathbb{R}^n_+$ is the point

$$x^*=(x_1,x_2,\dots,x_{n-1},-x_n)\in\mathbb{R}^n_-.$$

We will solve problem (2.16) for the half-space by setting

$$\Psi(x,y)=\Gamma(x^*,y)=\Gamma(x_1,\dots,x_{n-1},-x_n,y).$$

The idea is that the corrector $\Psi(x,y)$ is built from Γ by reflecting the singularity from $x\in\mathbb{R}^n_+$ to $x^*\in\mathbb{R}^n_-$. We note

$$\Psi(x,y)=\Gamma(x,y)\quad\text{on } y\in\partial\mathbb{R}^n_+,$$

and thus

$$\begin{cases}\Delta_y\Psi(x,y)=0 & \text{in } \mathbb{R}^n_+,\\ \Psi(x,y)=-\Gamma(x,y) & \text{on } \partial\mathbb{R}^n_+,\end{cases}$$

as required. Based on the above argument, we have the following proposition.

Proposition 2.24. *The Green's function for the half-space $\mathbb{R}^n_+$ is given by*

$$G(x,y)=\frac{1}{2\pi}\left(\log|x-y|-\log|x^*-y|\right),\qquad \textit{for } n=2,$$
$$G(x,y)=-\frac{1}{4\pi}\left(|x-y|^{-1}-|x^*-y|^{-1}\right),\qquad \textit{for } n=3.$$

Next we calculate the normal derivative of Green's function on the sphere.

Lemma 2.25. *Suppose G is the Green's function in $\mathbb{R}^n_+$. Then there holds*

$$\frac{\partial G}{\partial n}(x,y)=\frac{2x_n}{\omega_n}\frac{1}{|x-y|^n},\qquad \textit{for any } x\in\mathbb{R}^n_+ \textit{ and } y\in\partial\mathbb{R}^n_+.$$

Proof. We just consider the case $n = 3$. By direct computation,

$$\partial_{y_3} G(x,y) = -\frac{1}{4\pi}\left(\frac{x_3 - y_3}{|x-y|^3} - \frac{-x_3 - y_3}{|x^* - y|^3}\right) = -\frac{x_3}{2\pi}\frac{1}{|x-y|^3}.$$

Therefore

$$\frac{\partial G}{\partial n}(x,y) = -\partial_{y_3} G(x,y) = \frac{x_3}{2\pi}\frac{1}{|x-y|^3}, \quad \text{for any } x \in \mathbb{R}^3_+ \text{ and } y \in \partial\mathbb{R}^3_+.$$

□

Suppose now u solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = g & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.18}$$

Then from (2.17), we expect

$$u(x) = \frac{2x_n}{\omega_n}\int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\mathrm{d}y, \quad \text{for } x \in \mathbb{R}^n_+, \tag{2.19}$$

to be a representation formula for our solution. The function

$$K(x,y) = \frac{2x_n}{\omega_n}\frac{1}{|x-y|^n} \quad \text{for all } (x,y) \in \mathbb{R}^n_+ \times \partial\mathbb{R}^n_+,$$

is Poisson's kernel for $\mathbb{R}^n_+$, and (2.19) is Poisson's formula.
We must now check directly that formula (2.19) does indeed provide us with a solution of the Dirichlet Problem (2.18).

Theorem 2.26 (Poisson's formula for $\mathbb{R}^n_+$)**.** *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, and define u by (2.19). Then*

(i) $u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n_+)$,
(ii) $\Delta u = 0$ *in* $\mathbb{R}^n_+$,
(iii) $\lim\limits_{\substack{x\to x^0 \\ x\in\mathbb{R}^n_+}} u(x) = g(x^0)$ *for each point* $x^0 \in \partial\mathbb{R}^n_+$.

Proof. The proof is based on the property of $K(x,y)$ and we left it as an exercise. □

2.2.3. **Energy method**. Most of our analysis of harmonic functions thus far has depended upon fairly explicit representation formulas entailing the Green's functions. In this subsection, we will introduce some energy methods, which is to say techniques involving the L^2-norms of various expressions.

A. Uniqueness. Consider the Dirichlet Problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ \quad u(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

We have already employed the maximum principle in Theorem 2.9 to show uniqueness, but now we set forth a simple alternative proof. Assume Ω is open, bounded and $\partial\Omega$ is C^1.

Theorem 2.27 (Uniqueness)**.** *There exists at most one solution $u \in C^2(\bar{\Omega})$ of (2.20).*

Proof. Assume $\tilde{u}$ is another solution and set $\omega := u - \tilde{u}$. Then $\Delta\omega = 0$ in Ω, and so an integration by parts shows

$$0 = -\int_\Omega \omega\Delta\omega \mathrm{d}x = \int_\Omega |\nabla\omega|^2 \mathrm{d}x.$$

Thus $\nabla u \equiv 0$ in Ω and since $\omega = 0$ on $\partial\Omega$, we deduce $\omega = u - \tilde{u} = 0$ in Ω. □

B. Dirichlet's principle. Next let us demonstrate that a solution of the Dirichlet problem (2.20) can be characterized as the minimizer of an appropriate functional. For this, we define the energy functional

$$\mathcal{E}(\omega) := \int_\Omega \left(\frac{1}{2}|\nabla \omega|^2 - \omega f \right) \mathrm{d}x,$$

ω belonging to the admissible set

$$\mathcal{A} := \left\{ \omega \in C^2(\bar{\Omega}) | \omega = g \text{ on } \partial\Omega \right\}.$$

Theorem 2.28 (Dirichlet's principle)**.** *Assume $u \in C^2(\bar{\Omega})$ solves (2.20). Then*

$$\mathcal{E}(u) = \min_{\omega \in \mathcal{A}} \mathcal{E}(\omega). \tag{2.21}$$

Conversely, if $u \in \mathcal{A}$ satisfies (2.21), then u solves the boundary-value problem (2.20).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $\mathcal{E}(\cdot)$.

Proof. 1. For any $\omega \in \mathcal{A}$, from (2.20), we see that

$$\int_\Omega (-\Delta u - f)(u - \omega) \mathrm{d}x = 0.$$

By integration by parts and $u - \omega = 0$ on the boundary $\partial\Omega$, we have

$$\int_\Omega \nabla u \cdot \nabla(u - \omega) - f(u - \omega) \mathrm{d}x = 0,$$

which implies

$$\begin{aligned} \int_\Omega \left(|\nabla u|^2 - uf \right) \mathrm{d}x &= \int_\Omega \left(\nabla u \cdot \nabla \omega - \omega f \right) \mathrm{d}x \\ &\le \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \omega|^2 - \omega f \right) \mathrm{d}x. \end{aligned}$$

Rearranging, we obtain $\mathcal{E}(u) \le \mathcal{E}(\omega)$ for all $\omega \in \mathcal{A}$. Since $u \in \mathcal{A}$, we conclude $\mathcal{E}(u) = \min_{\omega \in \mathcal{A}} \mathcal{E}(\omega)$.

2. Conversely, suppose (2.21) holds. Fix any $v \in C_c^\infty(\Omega)$ and denote

$$f(s) := \mathcal{E}(u + sv) \quad \text{for all } s \in \mathbb{R}.$$

Since $u + sv \in \mathcal{A}$ for each s, we have

$$\min_{s \in \mathbb{R}} f(s) = f(0) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}s} f(0) = 0.$$

On the other hand, from the definition of $\mathcal{E}$, we see that

$$\begin{aligned} f(s) &= \int_\Omega \left(\frac{1}{2} \left| \nabla u + s \nabla v \right|^2 - (u + sv) f \right) \mathrm{d}x \\ &= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + s \nabla u \cdot \nabla v + \frac{s^2}{2}|\nabla v|^2 - (u + sv) f \right) \mathrm{d}x, \end{aligned}$$

which implies

$$0 = f'(0) = \int_\Omega \left(\nabla u \cdot \nabla v - fv \right) \mathrm{d}x = \int_\Omega \left(-\Delta u - f \right) v \mathrm{d}x.$$

This identity is valid for each function $v \in C_c^\infty(\Omega)$ and so $-\Delta u = f$ in Ω. □

2.3. **1D Heat equation.** Consider the 1D homogeneous heat equation

$$\partial_t u - \partial_x^2 u = 0, \quad \text{for } (t,x) \in [0,\infty) \times U, \tag{2.22}$$

and the 1D nonhomogeneous heat equation

$$\partial_t u - \partial_x^2 u = f, \quad \text{for } (t,x) \in [0,\infty) \times U, \tag{2.23}$$

where f is a regular function and U is $\mathbb{R}$ or half-line $\mathbb{R}^+$ or finite interval $[0, L]$.

2.3.1. **The initial value problem on** $\mathbb{R}$. Our first purpose in this subsection is to solve the problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & \text{in } \ (t,x) \in [0,\infty) \times \mathbb{R}, \\ u_{|t=0} = \phi(x), & \text{for } \ x \in \mathbb{R}. \end{cases} \tag{2.24}$$

The basic idea is to solve it for a particular ϕ and then build the general solution from this particular one. We first recall the following five invariance properties of the 1D heat equation (2.22):

(1) If $u(t,x)$ is a solution of (2.22), then $u(t, x-y)$ is also a solution of (2.22) for any $y \in \mathbb{R}$.
(2) If u is a solution of (2.22), then any derivative of u is also a solution of (2.22).
(3) A linear combination of solutions of (2.22) is also a solution of (2.22).
(4) An integral of solution is also a solution: if $S(t,x)$ is a solution of (2.22), then so is

$$v(t,x) = \int_{\mathbb{R}} S(t, x-y) g(y) \mathrm{d}y,$$

for any given function $g(y)$.
(5) If $u(t,x)$ is a solution of (2.22), then the dilated function $u_a(t,x) = u(at, \sqrt{a}x)$ is also a solution of (2.22) for any $a > 0$.

Our goal is to find a particular solution of (2.22) and then to construct all solutions (2.22) by property (4). Consider the particular solution $Q(t,x)$ for (2.22) with the following special initial data

$$Q(0,x) = 1 \text{ for } x > 0, \quad Q(0,x) = 0 \text{ for } x < 0. \tag{2.25}$$

Notice that the above initial data does not change under dilation. We will find the formula of $Q(t,x)$ in the following three steps.
Step 1. We look for $Q(t,x)$ having the special structure

$$Q(t,x) = g(s), \quad \text{where } s = \frac{x}{\sqrt{4t}}, \tag{2.26}$$

and the function $g : \mathbb{R} \to \mathbb{R}$ must be found. We expect Q to have the above special form since such structure is invariance under the dilation $x \to \sqrt{a}x$ and $t \to at$.
Step 2. Using (2.26) and the chain rule, we have

$$\begin{aligned} \partial_t Q &= \frac{\mathrm{d}g}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{1}{2t}\frac{x}{\sqrt{4t}} g'(s), \\ \partial_x Q &= \frac{\mathrm{d}g}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{\sqrt{4t}} g'(s), \quad \partial_x^2 Q = \frac{\mathrm{d}\partial_x Q}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{4t} g''(s). \end{aligned}$$

Combining the above identities with (2.22), we have

$$g''(s) + 2sg'(s) = 0 \Rightarrow Q(t,x) = g(s) = c_1 \int_0^s e^{-\rho^2} \mathrm{d}\rho + c_2 = c_1 \int_0^{\frac{x}{\sqrt{4t}}} e^{-\rho^2} \mathrm{d}\rho + c_2.$$

Step 3. We find the value of c_1 and c_2 from the initial data of Q. From the initial data (2.25), we see that

$$\text{Fix } x>0, \quad 1=\lim_{t\to 0+} Q(t,x)=c_1\int_0^{\infty} e^{-\rho^2}\mathrm{d}\rho + c_2 = c_1\frac{\sqrt{\pi}}{2}+c_2,$$
$$\text{Fix } x<0, \quad 0=\lim_{t\to 0+} Q(t,x)=c_1\int_0^{-\infty} e^{-\rho^2}\mathrm{d}\rho + c_2 = -c_1\frac{\sqrt{\pi}}{2}+c_2.$$

Based on the above identities, we have $c_1=\frac{1}{\sqrt{\pi}}$ and $c_2=\frac{1}{2}$ which implies

$$Q(t,x)=\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4t}}} e^{-\rho^2}\mathrm{d}\rho, \quad \text{for } t>0.$$

Having found $Q(t,x)$, we now consider

$$S(t,x)=\frac{\partial Q}{\partial x}(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \quad \text{for } t>0.$$

Definition 2.29. The function

$$S(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \quad \text{for } (t,x)\in(0,\infty)\times\mathbb{R},$$

is called the fundamental solution of the 1D heat equation.

For any $\phi:\mathbb{R}\to\mathbb{R}$, we also define

$$u(t,x)=\int_{\mathbb{R}} S(t,x-y)\phi(y)\mathrm{d}y, \quad \text{for } (t,x)\in(0,\infty)\times\mathbb{R}. \tag{2.27}$$

Theorem 2.30. *Assume $\phi\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$, and define u by (2.27). Then we have*

(i) $u\in C^\infty((0,\infty)\times\mathbb{R})$.
(ii) $\partial_t u(t,x)-\partial_x^2 u(t,x)=0$ *for* $(t,x)\in(0,\infty)\times\mathbb{R}$.
(iii) $\displaystyle\lim_{\substack{(t,x)\to(0,x^0)\\ t>0,\ x\in\mathbb{R}}} u(t,x)=\phi(x^0)$ *for each point* $x^0\in\mathbb{R}$.

Proof. Proof of (i) and (ii). Since the function $S(t,x)$ is infinitely differentiable, with uniformly bounded derivatives of all orders, on $[\delta,\infty)\times\mathbb{R}$ for each $\delta>0$, we see that the function $u\in C^\infty((0,\infty)\times\mathbb{R})$. Moreover, by an elementary computation,

$$\partial_t u(t,x)-\partial_x^2 u(t,x)=\int_{\mathbb{R}}\left(\partial_t S-\partial_x^2 S\right)(t,x-y)\phi(y)\mathrm{d}y=0,$$

since $S(t,x)$ is a solution for (2.22).
Proof of (iii). Fix $x^0\in\mathbb{R}$ and $\varepsilon>0$. Choose $\delta>0$ small enough such that

$$|\phi(y)-\phi(x^0)|<\varepsilon, \quad \text{for all } y\in(x^0-\delta,x^0+\delta). \tag{2.28}$$

For any $x\in(x^0-\frac{\delta}{2},x^0+\frac{\delta}{2})$, from $\int_{\mathbb{R}}S(t,\rho)\mathrm{d}\rho=1$, we have

$$\left|u(t,x)-\phi(x^0)\right|\le\int_{\mathbb{R}}S(t,x-y)|\phi(y)-\phi(x^0)|\mathrm{d}y\le\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3,$$

where

$$\mathcal{I}_1(t,x)=\int_{x^0-\delta}^{x^0+\delta}S(t,x-y)|\phi(y)-\phi(x^0)|\mathrm{d}y,$$
$$\mathcal{I}_2(t,x)=\int_{-\infty}^{x^0-\delta}S(t,x-y)|\phi(y)-\phi(x^0)|\mathrm{d}y,$$
$$\mathcal{I}_3(t,x)=\int_{x^0+\delta}^{\infty}S(t,x-y)|\phi(y)-\phi(x^0)|\mathrm{d}y.$$

From (2.28), we see that

$$\mathcal{I}_1(t,x) \le \varepsilon \int_{\mathbb{R}} S(t,x-y)\mathrm{d}y \le \varepsilon.$$

Then, we note that, for any $x \in (x^0 - \frac{\delta}{2}, x^0 + \frac{\delta}{2})$ and $y \le x^0 - \delta$, we have

$$|y-x^0| \le |x-y| + \frac{\delta}{2} \le |x-y| + \frac{1}{2}|y-x^0| \Rightarrow \frac{1}{2}|y-x^0| \le |y-x|.$$

Therefore, using the change of variable $z = y - x^0$ and $s = \frac{z}{\sqrt{t}}$, we have

$$\begin{aligned}
\mathcal{I}_2(t,x) &\le \frac{\|\phi\|_{L^\infty}}{\sqrt{4\pi t}} \int_{-\infty}^{x^0-\delta} e^{-\frac{|x-y|^2}{4t}}\mathrm{d}y \\
&\le \frac{\|\phi\|_{L^\infty}}{\sqrt{4\pi t}} \int_{-\infty}^{x^0-\delta} e^{-\frac{|y-x^0|^2}{16t}}\mathrm{d}y \\
&\le \frac{\|\phi\|_{L^\infty}}{\sqrt{4\pi t}} \int_{-\infty}^{-\delta} e^{-\frac{z^2}{16t}}\mathrm{d}z \le \frac{\|\phi\|_{L^\infty}}{\sqrt{4\pi}} \int_{-\infty}^{-\frac{\delta}{\sqrt{t}}} e^{-\frac{s^2}{16}}\mathrm{d}s \to 0, \quad \text{as } t \to 0^+.
\end{aligned}$$

Using a similar argument as above, we also have

$$\mathcal{I}_3(t,x) \to 0, \quad \text{as } t \to 0^+.$$

Combining the above estimates, we complete the proof of (iii). □

Notice that the continuity of ϕ was used only in only one part of the above proof. In actually, we can allow ϕ to have a jump discontinuity (See for example for $Q(0,x)$). Moreover, we can consider the piecewise continuity initial data.

Definition 2.31. A function $\phi : \mathbb{R} \to \mathbb{R}$ is said to have a jump at x_0 if both the limit of $\phi(x)$ as $x \to x_0$ from the right exists (denoted $\phi(x_0+)$) and the limit from the left exists (denoted $\phi(x_0-)$) but these two limits are not equal. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called piecewise continuous if in each finite interval it has only a finite number of jump and it is continuous at all other points.

Theorem 2.32. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded and piecewise continuous function. We define u by (2.27). Then we have*

(i) $u \in C^\infty((0,\infty)\times\mathbb{R})$.
(ii) $\partial_t u(t,x) - \partial_x^2 u(t,x) = 0$ *for* $(t,x) \in (0,\infty)\times\mathbb{R}$.
(iii) $\lim\limits_{\substack{(t,x)\to(0,x^0) \\ t>0,\ x\in\mathbb{R}}} u(t,x) = \frac{1}{2}(\phi(x_0+) + \phi(x_0-))$ *for each point* $x_0 \in \mathbb{R}$. *Ay every point of continuity, the limit equals* $\phi(x)$.

Proof. The proof is similar to Theorem 2.30 and we left it as an exercise. □

Remark 2.33. There are in fact infinitely many solutions of

$$\begin{cases} \partial_t u = \partial_x^2 u, & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u_{|t=0} = 0, & \text{for } x \in \mathbb{R}. \end{cases}$$

See for instance [2, Chapter 7.1]. Each of these solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$. The existence of such solutions tell us that we should not expect uniqueness (without any condition) for the solution of (2.24). However, maybe the conditional uniqueness (for example under a suitable condition of the growth rate) of the solution for (2.24) is true.

Our second purpose in this subsection is to solve the problem

$$\begin{cases} \partial_t u = \partial_x^2 u + f, & \text{in } \ (t,x) \in [0,\infty)\times\mathbb{R}, \\ u_{|t=0} = \phi(x), & \text{for } \ x \in \mathbb{R}. \end{cases} \tag{2.29}$$

We claim that the solution of (2.29) is

$$u(t,x) = \int_{\mathbb{R}} S(t,x-y)\phi(x)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} S(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s. \tag{2.30}$$

In actually, the definition of the above solution come from the Duhamels' principle which is a general method for obtaining solutions to inhomogeneous linear evolution equations like the heat equation, wave equation, and Transport equation. We start by a simplest ODE example:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + u(t) = f(t) \quad \text{with} \quad u(0) = \phi \in \mathbb{R}.$$

The unique solution for the above ODE is

$$u(t) = S(t)\phi + \int_0^t S(t-s)f(s)\mathrm{d}s \quad \text{where } S(t) = e^{-t}. \tag{2.31}$$

The first term in (2.31) represents the solution of the homogeneous ODE. The second term in (2.31) represents the solution of the inhomogeneous ODE with zero initial data, and it is the effect of the source term f.
Now, let us come back the 1D heat equation (2.29). Notice that there is an analogy between (2.30) and (2.31). More precisely, we denote $\mathcal{S}(t)$ by a operator

$$\left(\mathcal{S}(t)\phi\right)(x) = \int_{\mathbb{R}} S(t,x-y)\phi(y)\mathrm{d}y, \quad \text{for } t>0.$$

Then the formula (2.30) can be rewrite as

$$u(t) = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s)f(s)\mathrm{d}s, \quad \text{for } t>0.$$

Now we check (2.30) is a solution for (2.29).

Theorem 2.34. *Assume $f \in C^\infty([0,\infty)\times\mathbb{R})$ with compact support and $\phi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Define u by (2.30), then we have*

(i) $u \in C((0,\infty)\times\mathbb{R})$.
(ii) $\partial_t u(t,x) - \partial_x^2 u(t,x) = f(t,x)$ *for* $(t,x) \in (0,\infty)\times\mathbb{R}$.
(iii) $\lim\limits_{\substack{(t,x)\to(0,x^0)\\ t>0,\ x\in\mathbb{R}}} u(t,x) = \phi(x^0)$ *for each point* $x^0 \in \mathbb{R}$.

Proof. We decompose the function u in (2.30),

$$u(t,x) = u_1(t,x) + u_2(t,x), \quad \text{for } (t,x) \in (0,\infty)\times\mathbb{R},$$

where

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}} S(t,x-y)\phi(y)\mathrm{d}y, \\ u_2(t,x) &= \int_0^t \int_{\mathbb{R}} S(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s. \end{aligned}$$

From Theorem 2.30, we see that the function $u_1(t,x) \in C^\infty((0,\infty)\times\mathbb{R})$ and satisfy

$$\partial_t u_1(t,x) - \partial_x^2 u_1(t,x) = 0 \quad \text{and} \quad \lim_{\substack{(t,x)\to(0,x^0)\\ t>0,\ x\in\mathbb{R}}} u_1(t,x) = \phi(x^0).$$

Therefore, to complete the proof of Theorem 2.34, it is sufficient to show that $u_2 \in C^\infty((0,\infty)\times\mathbb{R})$ and satisfy

$$\partial_t u_2(t,x) - \partial_x^2 u_2(t,x) = f(t,x) \quad \text{and} \quad \lim_{\substack{(t,x)\to(0,x^0)\\ t>0,\ x\in\mathbb{R}}} u_2(t,x) = 0.$$

Since the fundamental solution $S(t,x)$ has a singularity at $t=0$, we cannot directly take the derivative of $S(t,x)$ and so we cannot take the derivative for the integral form. However, the function $f(t,x) \in C^\infty$ which means that we can directly take derivative of it. More precisely, we change variables $(s,y) \to (t-s, x-y)$ and rewrite

$$u_2(t,x) = \int_0^t \int_{\mathbb{R}} S(s,y) f(t-s, x-y) \mathrm{d}y \mathrm{d}s, \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}.$$

By an elementary computation, we know that,

$$\begin{aligned} \partial_x^m u_2(t,x) &= (-1)^m \int_0^t \int_{\mathbb{R}} S(s,y) \partial_y^m f(t-s,x-y) \mathrm{d}y \mathrm{d}s, \\ \partial_t u_2(t,x) &= -\int_0^t \int_{\mathbb{R}} S(s,y) \partial_s f(t-s,x-y) \mathrm{d}y \mathrm{d}s + \int_{\mathbb{R}} S(t,y) f(0,x-y) \mathrm{d}y. \end{aligned}$$

Using an induction argument, we have $u_2 \in C^\infty((0,\infty) \times \mathbb{R})$. Moreover, from the above two identities, we see that

$$\partial_t u_2(t,x) - \partial_x^2 u_2(t,x) = \mathcal{I}_3(t,x) + \mathcal{I}_4(t,x) + \mathcal{I}_5(t,x),$$

where

$$\begin{aligned} \mathcal{I}_3(t,x) &= \int_{\mathbb{R}} S(t,y) f(0,x-y) \mathrm{d}y, \\ \mathcal{I}_4(t,x) &= \int_0^\varepsilon \int_{\mathbb{R}} S(s,y)(\partial_t - \partial_x^2) f(t-s,x-y) \mathrm{d}y \mathrm{d}s, \\ \mathcal{I}_5(t,x) &= \int_\varepsilon^t \int_{\mathbb{R}} S(s,y)(-\partial_s - \partial_y^2) f(t-s,x-y) \mathrm{d}y \mathrm{d}s. \end{aligned}$$

Note that

$$\mathcal{I}_4(t,x) \le \left(\|\partial_t f\|_{L^\infty_{t,x}} + \|\partial_x^2 f\|_{L^\infty_{t,x}} \right) \int_0^\varepsilon \int_{\mathbb{R}} S(s,y) \mathrm{d}y \mathrm{d}s \le C_f \varepsilon.$$

Note also that, on the region $(s,y) \in (\varepsilon, t) \times \mathbb{R}$, the function $S(s,y)$ is smooth, and so we can use integrating by parts. More precisely, we have

$$\begin{aligned} \mathcal{I}_5(t,x) &= \int_\varepsilon^t \int_{\mathbb{R}} \left[\left(\partial_s - \partial_y^2 \right) S(s,y) \right] f(t-s,x-y) \mathrm{d}y \mathrm{d}s \\ &\quad + \int_{\mathbb{R}} S(\varepsilon,y) f(t-\varepsilon,x-y) \mathrm{d}y - \int_{\mathbb{R}} S(t,y) f(0,x-y) \mathrm{d}y \\ &= \int_{\mathbb{R}} S(\varepsilon,y) f(t-\varepsilon,x-y) \mathrm{d}y - \mathcal{I}_3(t,x), \end{aligned}$$

since the function $S(s,y)$ solves the heat equation. Combining the above estimate and identity, we see that

$$\partial_t u_2(t,x) - \partial_x^2 u_2(t,x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} S(\varepsilon,y) f(t-\varepsilon,x-y) \mathrm{d}y = f(t,x),$$

the limit as $\varepsilon \to 0$ being computed as in the proof of Theorem 2.30. Last, from the definition of $u_2(t,x)$, we have

$$\|u_2(t,x)\|_{L^\infty_x} \le \|f(t,x)\|_{L^\infty_{t,x}} \int_0^t \int_{\mathbb{R}} S(s,y) \mathrm{d}y \mathrm{d}s \lesssim t \|f(t,x)\|_{L^\infty_{t,x}} \to 0, \quad \text{as } t \to 0.$$

□

2.3.2. **The initial-boundary value problem on $\mathbb{R}^+$.** In this subsection, we fist study the Dirichlet problem of 1D heat equation on half-line $\mathbb{R}^+$. First, we consider

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x), & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ v(0,x) = \phi(x), & \text{for } t = 0, \\ v(t,0) = 0, & \text{for } x = 0. \end{cases} \tag{2.32}$$

We are looking for a solution formula analogous to (2.27). The general strategy is that we will reduce our problem to the old one. Note that the initial data $\phi(x)$ of (2.32) is defined only for $x \geq 0$ with $\phi(0) = 0$. Thus, we can consider the unique odd extension of ϕ to the whole line. That is

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ -\phi(-x) & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{2.33}$$

Let $u(t,x)$ be the solution of

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = \phi_{\text{odd}}(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.34}$$

According to §2.3.1, it is given by the formula

$$u(t,x) = \int_{\mathbb{R}} S(t,x-y)\phi(y)\mathrm{d}y, \quad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Consider the restriction of $u(t,x)$ on $\mathbb{R}^+ \times \mathbb{R}^+$,

$$v(t,x) := u(t,x), \quad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Since $u(t,x)$ is a solution of (2.34) on $\mathbb{R}^+ \times \mathbb{R}$ with odd initial data, $u(t,x)$ is also a odd function on $\mathbb{R}$ for any $t \in \mathbb{R}^+$ and so $u(t,0) = 0$ for all $t \in \mathbb{R}^+$. This implies that $v(t,0) = 0$ for all $t \in \mathbb{R}^+$ is also true. Furthermore, v solves the PDE as well as the initial condition for $x > 0$, simple because it is equal to u for $x > 0$ and u satisfies the same PDE for all x and the same initial condition for $x > 0$.
Now, we except to find the explicit formula for $v(t,x)$. First, from the formula for $u(t,x)$, $\phi_{\text{odd}}(x)$ is an odd function and change of variable, we have

$$\begin{aligned} u(t,x) &= \int_0^\infty S(t,x-y)\phi(y)\mathrm{d}y - \int_{-\infty}^0 S(t,x-y)\phi(-y)\mathrm{d}y \\ &= \int_0^\infty \left(S(t,x-t) - S(t,x+y)\right)\phi(y)\mathrm{d}y. \end{aligned}$$

Hence, for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) \phi(y)\mathrm{d}y.$$

We have just carried out the method of odd extensions or reflection method, so called because the graph of $\phi_{\text{odd}}(x)$ is the reflection of the graph of $\phi(x)$ across the origin.
Second, we consider the following Neumann problem

$$\begin{cases} \partial_t w(t,x) = \partial_x^2 w(t,x), & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ w(0,x) = \phi(x), & \text{for } t = 0, \\ \partial_x w(t,0) = 0, & \text{for } x = 0. \end{cases} \tag{2.35}$$

Using again the reflection method (consider an even extension), we obtain an explicit formula for $w(t,x)$:

$$w(t,x)=\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{(x-y)^2}{4t}}+e^{-\frac{(x+y)^2}{4t}}\right)\phi(y)\mathrm{d}y.$$

Last, we consider the 1D inhomogeneous heat equation on half-line $\mathbb{R}^+$. Consider

$$\begin{cases}\partial_t v(t,x)=\partial_x^2 v(t,x)+f(t,x), & \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^+,\\ v(0,x)=\phi(x), & \text{for } t=0,\\ v(t,0)=0, & \text{for } x=0.\end{cases}\tag{2.36}$$

Using again the reflection method (consider an odd extension), we obtain an explicit formula for $v(t,x)$:

$$\begin{aligned}v(t,x)=&\int_0^\infty\left(S(t,x-y)-S(t,x+y)\right)\phi(y)\mathrm{d}y\\&+\int_0^t\int_0^\infty\left(S(t-s,x-y)-S(t-s,x+y)\right)f(s,y)\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Now we consider the more complicated problem of a boundary source $h(t)$ on the half-line; that is,

$$\begin{cases}\partial_t v(t,x)=\partial_x^2 v(t,x)+f(t,x), & \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^+,\\ v(0,x)=\phi(x), & \text{for } t=0,\\ v(t,0)=h(t), & \text{for } x=0.\end{cases}\tag{2.37}$$

We may use the following subtraction device to reduce (2.37) to (2.36). Consider an auxiliary function $V(t,x)=v(t,x)-h(t)$. Then $V(t,x)$ satisfy

$$\begin{cases}\partial_t V(t,x)=\partial_x^2 V(t,x)+f(t,x)-h'(t), & \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^+,\\ V(0,x)=\phi(x)-h(0), & \text{for } t=0,\\ V(t,0)=0, & \text{for } x=0.\end{cases}$$

We have known that how to find an explicit formula for $V(t,x)$ and once we find $V(t,x)$, we recover v by $v(t,x)=V(t,x)-h(t)$.

2.3.3. **Maximum Principle**. In this subsection, we study the qualitative property of the 1D heat equation in a rectangle $(t,x)\in D_{T,\ell}=[0,T]\times(0,\ell)$. More precisely, we consider

$$\partial_t u-\partial_x^2 u=f(t,x),\quad \text{for } (t,x)\in D_{T,\ell}=[0,T]\times(0,\ell).\tag{2.38}$$

We start by the maximum principle for 1D heat equation.

Theorem 2.35. *Suppose $u=u(t,x)$ is the solution to the 1D heat equation (2.38). Assume that the source term $f=f(t,x)$ is nowhere positive: $f(t,x)\le 0$ for all $(t,x)\in D_{T,\ell}$. Then the maximum of $u(t,x)$ on the closed rectangle $\bar{D}_{T,\ell}$ is attained at $t=0$ or $x=0$ or $x=\ell$.*

Proof. First, let us to prove the Theorem under the stronger assumption $f(t,x)<0$ in $D_{T,\ell}$ which implies

$$\partial_t u(t,x)<\partial_x^2 u(t,x)\quad \text{in } D_{T,\ell}.\tag{2.39}$$

Suppose first that $u(t,x)$ has a local maximum at a point (t_0,x_0) in the $\mathrm{Int}(\mathrm{D}_{\mathrm{T},\ell})$. Then we have

$$\partial_t u(t_0,x_0)=\partial_x u(t_0,x_0)=0.\tag{2.40}$$

Our assumption implies that the scalar function $h(x)=u(t_0,x)$ has a maximum at $x=x_0$. Thus, by the second derivative test for functions of a single variable,

$$h''(x_0)=\partial_x^2 u(t_0,x_0)\le 0.\tag{2.41}$$

However, the requirements (2.40) and (2.41) are clearly incompatible with the inequality (2.41). In conclusion, the solution $u(t,x)$ cannot have a local maximum at any point in the $\mathrm{Int}(\mathrm{D}_{\mathrm{T},\ell})$. Then, we suppose that $u(t,x)$ has a local maximum at a point (T,x_0) for $x_0 \in (0,\ell)$. Note that, our assumption implies that the scalar function $g(t) = u(t,x_0)$ would be nondecreasing at $t = T$, and hence $g'(T) = \partial_t u(T,x_0) \geq 0$. The preceding argument also implies that $\partial_x^2 u(T,x_0) \leq 0$ and again these two requirements are incompatible with (2.39). We conclude that any (local) maximum must attend at one of the other three sides of the rectangle $\bar{D}_{T,\ell}$, in accordance with the statement of the theorem.
Second, we expect to generalize the argument to the case $f(t,x) \leq 0$ and this will require a little trick. Starting with the solution $u(t,x)$ to (2.38), we set

$$v_\varepsilon(t,x) = u(t,x) + \varepsilon x^2, \quad \text{where } \varepsilon > 0.$$

Then by direct computation, we have

$$\partial_t v(t,x) - \partial_x^2 v(t,x) = \tilde{f}(t,x) \quad \text{where } \tilde{f}(t,x) = f(t,x) - 2\varepsilon < 0.$$

Thus, by the previous argument, a local maximum of $v(t,x)$ can attend only at $t = 0$ or $x = 0$ or $x = \ell$. Let M denote the maximum value of $u(t,x)$ on the indicated three sides of the rectangle. Then we have the maximum value of $v(t,x)$ on the indicated three sides of the rectangle is smaller than $M + \varepsilon\ell^2$. Therefore

$$\max_{(t,x)\in\bar{D}_{T,\ell}} v(t,x) \leq M + \varepsilon\ell^2 \Rightarrow \max_{(t,x)\in\bar{D}_{T,\ell}} u(t,x) \leq M + \varepsilon\ell^2.$$

Let $\varepsilon \to 0^+$ in the above inequality, we complete the proof of Theorem 2.35. □

Consider the Dirichlet problem for the 1D heat equation:

$$\begin{cases} \partial_t u - \partial_x^2 u = f(t,x), & \text{for } (t,x) \in (0,\infty)\times(0,\ell), \\ u(0,x) = \phi(x), & \text{for } x \in \mathbb{R}, \\ u(t,0) = g(t), & \text{for } t \in (0,\infty), \\ u(t,\ell) = h(t), & \text{for } t \in (0,\infty). \end{cases} \tag{2.42}$$

The maximum principle can be used to give a proof of uniqueness for the Dirichlet problem (2.42).

Theorem 2.36. *There is at most one solution of* (2.42).

Proof. Let $u_1(t,x)$ and $u_2(t,x)$ be two solutions of (2.42). Let $w(t,x) = u_1(t,x) - u_2(t,x)$ be their difference. Then we have

$$\begin{cases} \partial_t w - \partial_x^2 w = 0, & \text{for } (t,x) \in (0,\infty)\times(0,\ell), \\ w(0,x) = 0, & \text{for } x \in \mathbb{R}, \\ w(t,0) = 0, & \text{for } t \in (0,\infty), \\ w(t,\ell) = 0, & \text{for } t \in (0,\infty). \end{cases}$$

Fix $T > 0$. By the maximum principle, we see that $w(t,x) \leq 0$ on $\bar{D}_{T,\ell}$ and the same for the minimum shows that $w(t,x) \geq 0$ on $\bar{D}_{T,\ell}$. Therefore, $w(t,x) = 0$ on $\bar{D}_{T,\ell}$ which implies that $w(t,x) = 0$ on $(0,\infty)\times(0,\ell)$ (since the arbitrary choice of T). The proof of Theorem 2.36 is complete. □

Next, we introduce a second proof of uniqueness of problem (2.42) by the energy method.

Proof of Theorem 2.36. Multiplying the equation for $w = u_1 - u_2$ by w itself, we can get

$$0 = (\partial_t w - \partial_x^2 w)w = \frac{1}{2}\partial_t(w^2) - \partial_x(w\partial_x w) + (\partial_x w)^2.$$

Integrating the above identity on the interval $[0,\ell]$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell (w(t,x))^2\mathrm{d}x = -\int_0^\ell (\partial_x w(t,x))^2\mathrm{d}x \le 0.$$

Therefore the L^2 norm of w is decreasing and so

$$\int_0^\ell (w(t,x))^2\mathrm{d}x \le \int_0^\ell (w(0,x))^2\mathrm{d}x = 0 \Rightarrow w(t,x)=0, \quad \text{for all } (t,x)\in(0,\infty)\times\mathbb{R},$$

which means that $u_1(t,x)=u_2(t,x)$ for all $(t,x)\in(0,\infty)\times\mathbb{R}$. □

We finish this subsection by the stability of the (2.42) in two different senses. Consider the following two Dirichlet problem

$$\begin{cases} \partial_t u_1 - \partial_x^2 u_1 = f(t,x), & \text{for } (t,x)\in(0,\infty)\times(0,\ell),\\ u_1(0,x)=\phi_1(x), & \text{for } x\in(0,\ell),\\ u_1(t,0)=g(t), & \text{for } t\in(0,\infty),\\ u_1(t,\ell)=h(t), & \text{for } t\in(0,\infty), \end{cases} \tag{2.43}$$

as well as

$$\begin{cases} \partial_t u_2 - \partial_x^2 u_2 = f(t,x), & \text{for } (t,x)\in(0,\infty)\times(0,\ell),\\ u_2(0,x)=\phi_2(x), & \text{for } x\in(0,\ell),\\ u_2(t,0)=g(t), & \text{for } t\in(0,\infty),\\ u(t,\ell)=h(t), & \text{for } t\in(0,\infty). \end{cases} \tag{2.44}$$

Note that, the above two problems with the same source term f, boundary conditions $g(t)$ and $h(t)$, but with different initial data ϕ_1 and ϕ_2. Let $w(t,x)=u_1(t,x)-u_2(t,x)$. Using the energy argument as above for w, we have

$$\int_0^\ell (w(t,x))^2\mathrm{d}x \le \int_0^\ell (w(0,x))^2\,\mathrm{d}x, \quad \text{for all } t>0,$$

which means that

$$\int_0^\ell (u_1(t,x)-u_2(t,x))^2\mathrm{d}x \le \int_0^\ell (\phi_1(x)-\phi_2(x))^2\,\mathrm{d}x, \quad \text{for all } t>0, \tag{2.45}$$

The inequality (2.45) means the stability of (2.42) in the L^2 norm sense. On the other hand, using the maximum principle (both with minimum principle) for w, we see that

$$\sup_{t\ge0}\max_{x\in[0,\ell]}|u_1(t,x)-u_2(t,x)| \le \max_{x\in[0,\ell]}|\phi_1(x)-\phi_2(x)|. \tag{2.46}$$

The inequality (2.46) means the stability of (2.42) in the uniform sense.

2.4. **1D Wave equation.** Consider the 1D homogeneous wave equation

$$\partial_t^2 u - \partial_x^2 u = 0, \quad \text{for } (t,x)\in[0,\infty)\times U, \tag{2.47}$$

and the 1D nonhomogeneous wave equation

$$\partial_t^2 u - \partial_x^2 u = f, \quad \text{for } (t,x)\in[0,\infty)\times U, \tag{2.48}$$

where f is a regular function and U is $\mathbb{R}$ or half-line $\mathbb{R}^+$ or finite interval $[0,L]$.

2.4.1. **The initial value problem on** $\mathbb{R}$. Our first purpose in this subsection is to solve the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1), & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.49}$$

Note that the PDE in (2.49) can be factored to read

$$(\partial_t + \partial_x)(\partial_t u - \partial_x u) = 0.$$

We denote $v(t,x) = \partial_t u(t,x) - \partial_x u(t,x)$ and so

$$\partial_t v(t,x) + \partial_x v(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

This is a transport equation with constant coefficients. Using the formula of the solution for transport equation, we have

$$v(t,x) = a(x-t) \quad \text{and} \quad a(x) := v(0,x).$$

From the definition of $v(t,x)$, we also have

$$\partial_t u(t,x) - \partial_x u(t,x) = a(x-t), \quad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Note that this is an inhomogeneous transport equation and so we have

$$\begin{aligned} u(t,x) &= \int_0^t a(x + (t-s) - s)\mathrm{d}s + b(x+t) \\ &= \frac{1}{2}\int_{x-t}^{x+t} a(y)\mathrm{d}y + b(x+t) \quad \text{where} \quad b(x) := u(0,x) = u_0(x). \end{aligned}$$

Recall that,

$$a(x) = v(0,x) = \partial_t u(0,x) - \partial_x u(0,x) = u_1(x) - \partial_x u_0(x), \quad \text{for } x \in \mathbb{R}.$$

Therefore, combining the above identities, we have

$$u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\mathrm{d}y. \tag{2.50}$$

This is d'Alembert's formula. We have derived formula (2.50) assuming u is a smooth solution of (2.49). We need to check that this really is a solution.

Theorem 2.37. *Assume $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$, and define u by d'Alembert's formula* (2.50)*. Then*

(i) $u \in C^2(\mathbb{R}^+ \times \mathbb{R})$,
(ii) $\partial_t^2 u - \partial_x^2 u = 0$ *in* $\mathbb{R}^+ \times \mathbb{R}$,
(iii) $\lim_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0)$, $\lim_{(t,x)\to(0,x_0)} \partial_t u(t,x) = u_1(x_0)$, *for all* $x_0 \in \mathbb{R}$.

Proof. The proof is a straightforward calculation, and we let it as an exercise. □

Remark 2.38. In view of (2.50), the solution u has the form

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left(u_0(t+x) + \int_0^{x+t} u_1(y)\mathrm{d}y\right) \\ &\quad + \frac{1}{2}\left(u_0(t-x) + \int_{x-t}^{0} u_1(y)\mathrm{d}y\right) = F(t+x) + G(t-x), \end{aligned}$$

for appropriate functions F and G. Conversely any function of this form solves $\partial_t^2 u - \partial_x^2 u = 0$. Hence the general solution of the 1D wave equation is a sum of the general solution of $\partial_t v - \partial_x v = 0$ and the general solution of $\partial_t v + \partial_x v = 0$.

Our second purpose in this subsection is to solve the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f, & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1), & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.51}$$

We claim that the solution of (2.51) is

$$\begin{aligned} u(t,x) = {} & \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\mathrm{d}y \\ & + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{2.52}$$

Here, we need use again the Duhamel's principle. Heuristically, we consider an analogized second-order ODE

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) + u(t) = f(t), \quad u(0) = u_0, \quad \partial_t u(0) = u_1. \tag{2.53}$$

The solution of (2.53) is

$$u(t) = S'(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)\mathrm{d}s, \quad \text{where } S(t) = \cos t. \tag{2.54}$$

The key to understanding formula (2.54) is that $S(t)u_1$ is the solution of problem (2.53) in the case that $u_0 = f(t) = 0$. Let us return to the PDE (2.51). The basic solution operator should be given by the u_1 term. That is

$$\mathcal{S}(t)u_1 = \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\mathrm{d}y, \quad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Note that $\mathcal{S}(t)u_1$ solves $\partial_t^2 u - \partial_x^2 u = 0$, $u(0,x) = 0$ and $\partial_t u(0,x) = u_1$. $\mathcal{S}(t)$ is the source operator or solution operator. By (2.54), we expect the u_0 term to be $\frac{\partial}{\partial t}\mathcal{S}(t)u_0$. In fact, we have

$$\frac{\partial}{\partial t}\mathcal{S}(t)u_0 = \frac{1}{2}\frac{\partial}{\partial t}\int_{x-t}^{x+t} u_0(y)\mathrm{d}y = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right).$$

Let us now take the source term f; that is, $u_0 = u_1 = 0$. By analogy with the last term in (2.31), the solution should be

$$u(t) = \int_0^t \mathcal{S}(t-s)f(s)\mathrm{d}s = \int_0^t \int_{x-t}^{x+t} f(s,y)\mathrm{d}y\mathrm{d}s,$$

which is the last term in (2.52). We need to check that (2.52) is a solution to (2.51).

Theorem 2.39. *Assume* $(u_0, u_1, f) \in C^2(\mathbb{R}) \times C^1(\mathbb{R}) \times C^1(\mathbb{R}^2)$, *and define* u *by* (2.52). *Then*

(i) $u \in C^2(\mathbb{R}^+ \times \mathbb{R})$,
(ii) $\partial_t^2 u - \partial_x^2 u = f$ in $\mathbb{R}^+ \times \mathbb{R}$,
(iii) $\lim\limits_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0)$, $\lim\limits_{(t,x)\to(0,x_0)} \partial_t u(t,x) = u_1(x_0)$, *for all* $x_0 \in \mathbb{R}$.

Proof. The proof is based on a not easy calculation, and we let it as an exercise. □

2.4.2. **The initial-boundary value problem on** $\mathbb{R}^+$. In this subsection, we consider the following initial-boundary value problem

$$\begin{cases} \partial_t^2 v(t,x) = \partial_x^2 v(t,x), & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (v, \partial_t v)_{|t=0} = (u_0, u_1), \ v(t,0) = 0 & \text{for } t \in \mathbb{R}^+. \end{cases} \tag{2.55}$$

The reflection method is carried out in the same way as in §2.3.2. Consider the odd extension of both of the initial data to the whole line $\mathbb{R}$, $u_{0,\mathrm{odd}}$ and $u_{1,\mathrm{odd}}$. Let $u(t,x)$ be the solution of the initial value problem on $\mathbb{R}$ with the initial data

$u_{0,\mathrm{odd}}$ and $u_{1,\mathrm{odd}}$. Then, $u(t,x)$ is once again an odd function of x and so we have $u(t,0)=0$. Define $v(t,x)=u(t,x)$ for $(t,x)\in\mathbb{R}^+\times\mathbb{R}^+$. Then $v(t,x)$ is precisely the solution we are looking for. From the formula (2.50), we have for $x\geq 0$,

$$v(t,x)=u(t,x)=\frac{1}{2}\left[u_{0,\mathrm{odd}}(x+t)+u_{0,\mathrm{odd}}(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}u_{1,\mathrm{odd}}(y)\mathrm{d}y.$$

We split the spacetime region $\mathbb{R}^+\times\mathbb{R}^+$ into two parts $A=\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^+ : x>t>0\}$ and $B=\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^+ : t>x>0\}$. First, we notice that, for any $(t,x)\in A$ only positive arguments occur in the formula, so that $v(t,x)$ is given by the usual formula:

$$v(t,x)=\frac{1}{2}\left(u_0(x+t)+u_0(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t}u_1(y)\mathrm{d}y,\quad \text{for } (t,x)\in A.$$

Second, we notice that, for $(t,x)\in B$, we have $u_{0,\mathrm{odd}}(x-t)=-u_0(t-x)$, and so

$$\begin{aligned}v(t,x)=&\frac{1}{2}\left(u_0(t+x)-u_0(t-x)\right)\\&+\frac{1}{2}\int_0^{x+t}u_1(y)\mathrm{d}y+\frac{1}{2}\int_{x-t}^{0}(-u_1(-y))\mathrm{d}y\\=&\frac{1}{2}\left(u_0(t+x)-u_0(t-x)\right)+\frac{1}{2}\int_{t-x}^{t+x}u_1(y)\mathrm{d}y.\end{aligned}$$

The complete solution is given

$$v(t,x)=\begin{cases}\dfrac{1}{2}\left(u_0(x+t)+u_0(x-t)\right)+\dfrac{1}{2}\displaystyle\int_{x-t}^{x+t}u_1(y)\mathrm{d}y, & \text{for } (t,x)\in A\\ \dfrac{1}{2}\left(u_0(t+x)-u_0(t-x)\right)+\dfrac{1}{2}\displaystyle\int_{t-x}^{t+x}u_1(y)\mathrm{d}y, & \text{for } (t,x)\in B.\end{cases}\tag{2.56}$$

Note that, our solution does not belong to C^2, unless $u_0''(0)=0$.

2.4.3. **The initial value problem on** $\mathbb{R}^3$. In this subsection, we consider the following initial value problem for 3D wave equation,

$$\begin{cases}\partial_t^2u-\Delta u=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R},\\ (u,\partial_t u)_{|t=0}=(u_0,u_1), & x\in\mathbb{R}.\end{cases}\tag{2.57}$$

We intend to derive an explicit formula for u in terms of u_0 and u_1. The plan will be to study first the average of u over certain spheres. These averages, taken as functions of the time t and the radius r, turn out to solve the so called Euler-Poisson-Darboux equation, a PDE which we can convert into the 1D wave equation. Applying d'Alembert's formula (2.50), we could obtain a explicit formula for solution of Euler-Poisson-Darboux and then conclude the formula for solution of (2.57).

Definition 2.40. (i) Let $x\in\mathbb{R}^3$, $t>0$ and $r>0$. Define

$$\mathcal{U}(t,x,r):=\frac{1}{4\pi r^2}\int_{\partial B_r(x)}u(t,y)\mathrm{d}S_y,\tag{2.58}$$

the average of $u(t,x)$ over the sphere $\partial B_r(x)$.
(ii) Similarly,

$$\begin{cases}\mathcal{U}_0(x,r):=\dfrac{1}{4\pi r^2}\displaystyle\int_{\partial B_r(x)}u_0(y)\mathrm{d}S_y,\\ \mathcal{U}_1(x,r):=\dfrac{1}{4\pi r^2}\displaystyle\int_{\partial B_r(x)}u_1(y)\mathrm{d}S_y.\end{cases}\tag{2.59}$$

For fixed x, we hereafter regard $\mathcal{U}$ as a function of (t,r) and discover a PDE that $\mathcal{U}$ solves.

Lemma 2.41 (Euler-Poisson-Darboux equation)**.** *Fix $x \in \mathbb{R}^3$ and let $u \in C^2(\mathbb{R}^+ \times \mathbb{R}^3)$ satisfy (2.57). Then $\mathcal{U} \in C^2(\mathbb{R}^+ \times \mathbb{R}^+)$ and*

$$\begin{cases} \partial_t^2 \mathcal{U} - \partial_r^2 \mathcal{U} - \dfrac{2}{r}\mathcal{U} = 0, & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (\mathcal{U}, \partial_t \mathcal{U})_{|t=0} = (\mathcal{U}_0, \mathcal{U}_1), & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.60}$$

Proof. The regularity of $\mathcal{U}$ and $(\mathcal{U}, \partial_t \mathcal{U})_{|t=0} = (\mathcal{U}_0, \mathcal{U}_1)$ are consequence of (2.57). Now we prove the Euler-Poisson-Darboux equation for $\mathcal{U}$. First, we rewrite

$$\begin{aligned} \mathcal{U}(t,x,r) &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} u(t,y) \mathrm{d}S_y \\ &= \frac{1}{4\pi} \int_{\partial B_1(0)} u(t, x + rz) \mathrm{d}S_z \end{aligned}$$

Therefore, from (2.57), we have

$$\begin{aligned} \partial_r \mathcal{U}(t,x,r) &= \frac{1}{4\pi} \int_{\partial B_1(0)} z \cdot \nabla_x u(t, x+rz) \mathrm{d}S_z \\ &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} \nabla u(t,y) \cdot \frac{y-x}{r} \mathrm{d}S_y \\ &= \frac{1}{4\pi r^2} \int_{B_r(x)} \Delta u(t,y) \mathrm{d}y = \frac{1}{4\pi r^2} \int_{B_r(x)} \partial_t^2 u(t,y) \mathrm{d}y, \end{aligned}$$

which implies

$$\partial_r \left(r^2 \partial_r \mathcal{U} \right) = \frac{1}{4\pi} \int_{\partial B_r(x)} \partial_t^2 u(t,y) \mathrm{d}S_y = r^2 \partial_t^2 \mathcal{U}.$$

From the above identity, we obtain the Euler-Poisson-Darboux equation for $\mathcal{U}$. $\square$

We introduce

$$\widetilde{\mathcal{U}} = r\mathcal{U}, \quad \widetilde{\mathcal{U}}_0 = r\mathcal{U}_0, \quad \widetilde{\mathcal{U}}_1 = r\mathcal{U}_1.$$

From Lemma 2.41, we see that

$$\begin{cases} \partial_t^2 \widetilde{\mathcal{U}} - \partial_r^2 \widetilde{\mathcal{U}} = 0, & \text{for } (t,r) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ \widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_0, \ \partial_t \widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_1, & \text{for } (t,r) \in \{0\} \times \mathbb{R}^+, \\ \widetilde{\mathcal{U}} = 0, & \text{for } (t,r) \in \mathbb{R}^+ \times \{0\}. \end{cases} \tag{2.61}$$

Then, using (2.56), we have

$$\widetilde{\mathcal{U}}(t,x,r) = \frac{1}{2} \left(\widetilde{\mathcal{U}}_0(t+r) - \widetilde{\mathcal{U}}_0(t-r) \right) + \frac{1}{2} \int_{t-r}^{t+r} \widetilde{\mathcal{U}}_1(\rho) \mathrm{d}\rho.$$

From the definition of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, we see that

$$\begin{aligned} u(t,x) &= \lim_{r \to 0^+} \left(r^{-1} \widetilde{\mathcal{U}}(t,x,r) \right) \\ &= \frac{1}{2} \lim_{r \to 0^+} \left(r^{-1} \left(\widetilde{\mathcal{U}}_0(t+r) - \widetilde{\mathcal{U}}_0(t-r) \right) + r^{-1} \int_{t-r}^{t+r} \widetilde{\mathcal{U}}_1(\rho) \mathrm{d}\rho \right) = \widetilde{\mathcal{U}}_0'(t) + \widetilde{\mathcal{U}}_1(t). \end{aligned}$$

Based on a change of variable and elementary computation, we have

$$\widetilde{\mathcal{U}}_0'(t) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \left(u_0(y) + \nabla u_0(y) \cdot (y-x) \right) \mathrm{d}S_y,$$

which implies

$$u(t,x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \left(u_0(y) + \nabla u_0(y) \cdot (y-x) + t u_1(y) \right) \mathrm{d}S_y.$$

This is Kirchhoff's formula for the solution of the initial-value problem (2.57).

3. Boundary Problems

In this section, we will introduce the method of separation of variables. In mathematics, separation of variables (also known as the Fourier method) is any of several methods for solving ordinary and partial differential equations, in which algebra allows one to rewrite an equation so that each of two variables occurs on a different side of the equation.
In this section, the presentation is usually close to [4, Chapter 4].

3.1. **Separation of variables, the Dirichlet condition.** We first consider the homogeneous Dirichlet conditions for the 1D homogeneous wave equation:

$$\begin{cases} \partial_t^2 u = \partial_x^2 u, & \text{for } (t,x) \in [0,\infty) \times (0,\ell), \\ u(t,0) = 0 = u(t,\ell), & \text{for } t \in [0,\infty), \end{cases} \tag{3.1}$$

with some initial conditions

$$(u, \partial_t u)_{|t=0} = (\phi, \psi), \quad \text{for } x \in (0,\ell). \tag{3.2}$$

The method we shall use consists of building up the general solution as a linear combination of special ones that are easy to find. More specifically, a separated solution is a solution of (3.1)- (3.2) of the form

$$u(t,x) = T(t)X(x), \quad \text{for } (t,x) \in [0,\infty) \times (0,\ell). \tag{3.3}$$

Our goal is to look for as many separated solutions as possible.
Plugging the form (3.3) into the wave equation (3.1), we get

$$X(x)T''(t) = X''(x)T(t), \quad \text{for } (t,x) \in [0,\infty) \times (0,\ell),$$

and so, dividing by $-XT$, we find

$$-\frac{T''}{T} = -\frac{X''}{X} = \lambda, \quad \text{for } (t,x) \in [0,\infty) \times (0,\ell). \tag{3.4}$$

This defines a quantity λ which must be a constant (Since we can argue that λ does not depend on x because of the first expression and does not depend on t because of the second expression, so that it does not depend on any variable).
Case I. Let $\lambda > 0$. We denote $\lambda = \beta^2$ where $\beta > 0$. Then the equations (3.4) are a pair of separate ODE for X and T:

$$X'' + \beta X = 0 \quad \text{and} \quad T'' + \beta T = 0. \tag{3.5}$$

The solutions of (3.5) have the form

$$\begin{aligned} X(x) &= C\cos\beta x + D\sin\beta x, \\ T(t) &= A\cos\beta t + B\sin\beta t, \end{aligned} \tag{3.6}$$

where A, B, C and D are constants.
The second step is to impose the boundary conditions (3.1) on the separated solution. They simply require that $X(0) = X(\ell) = 0$. Thus

$$0 = X(0) = C \quad \text{and} \quad 0 = X(\ell) = D\sin\beta\ell.$$

Surely we are not interested in the obvious solution $C = D = 0$. So we must have $\beta\ell = n\pi$, a root of the sine function. That is,

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{\ell}, \quad \text{for } n \in \mathbb{N}^+,$$

are distinct solutions, Each sine function may be multiplied by an arbitrary constant. Therefore, there are an infinite number of separated solutions of (3.1), one for each n. They are

$$u_n(t,x) = \left(A_n\cos\frac{n\pi t}{\ell} + B_n\sin\frac{n\pi t}{\ell}\right)\sin\frac{n\pi x}{\ell}, \quad \text{for } n \in \mathbb{N}^+,$$

where A_n and B_n are arbitrary constants. The sum of solutions is again a solution, so any finite sum

$$u(t,x) = \sum_{n=1}^{N} \left(A_n \cos \frac{n\pi t}{\ell} + B_n \sin \frac{n\pi t}{\ell} \right) \sin \frac{n\pi x}{\ell},$$

is also a solution of (3.1). The above formula solves (3.1)-(3.2), provided that

$$\phi(x) = \sum_{n=1}^{N} A_n \sin \frac{n\pi x}{\ell} \quad \text{and} \quad \psi(x) = \sum_{n=1}^{N} \frac{n\pi}{\ell} B_n \sin \frac{n\pi x}{\ell}. \tag{3.7}$$

Thus for any initial data of this form, the problem (3.1)-(3.2) has a simple explicit solution. But such initial data (3.7) clearly are very special. So let us try to take infinite sums. Then we ask what kind of initial data paris (ϕ, ψ) can be expanded as in (3.7) for some choice of coefficients A_n and B_n? This question was the source of great disputes for half a century around 1800, but the final result of the disputes was very simple: Practically any function ϕ on the interval $(0, \ell)$ can be expanded in an infinite series (3.7). We will show this in Section 4.

Case II. Let $\lambda = 0$. This would mean that $X'' = 0$, so that $X(x) = C + Dx$. But $X(0) = X(\ell) = 0$ implies that $C = D = 0$, so that $X(x) \equiv 0$. Therefore, $\lambda = 0$ is not an eigenvalue.

Case III. Let $\lambda = -\gamma^2 < 0$. Then $X'' = \gamma^2 X$, so that $X(x) = C \cosh \gamma x + D \sinh \gamma x$. Then $0 = X(0) = C$ and $0 = X(\ell) = D \sinh \gamma \ell$. Hence $D = 0$ since $\sinh \gamma \ell \neq 0$. Therefore, $\lambda < 0$ is not an eigenvalue.

In conclusion, the only eigenvalues λ of our problem (3.4) are positive numbers; in fact, they are $\left\{ \left(\frac{n\pi}{\ell}\right)^2 \right\}_{n \in \mathbb{N}}$.

The analogous problem for heat equation is

$$\begin{cases} \partial_t u = \partial_x^2 u, & \text{for } (t,x) \in [0,\infty) \times (0,\ell), \\ u(t,0) = u(t,\ell) = 0, & \text{for } t \in [0,\infty), \\ u(0,x) = \phi(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{3.8}$$

To solve it, we separate the variables $u = T(t)X(x)$ as before. This time we get

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{costant}.$$

Therefore, $T(t)$ satisfies the equation $T' = -\lambda T$, whose solution is $T(t) = Ae^{-\lambda t}$. Furthermore, we have

$$-X'' = \lambda X \text{ in } x \in (0,\ell) \quad \text{with } X(0) = X(\ell) = 0.$$

This is precisely the same problem for $X(x)$ as before and so has the same solutions. Because of the form of $T(t)$, we have

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 t} \sin \frac{n\pi x}{\ell},$$

is the solution of (3.8) provided that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}.$$

Once again, our solution is expressible for each t as a Fourier sine series in x provided that the initial data are.

3.2. **The Neumann condition.** The same method works for both the Neumann and Robin boundary conditions. In the former case, the second line of (3.1) is replaced by $\partial_x u(t,0) = \partial_x u(t,\ell) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$-X'' = \lambda X, \quad X'(0) = X'(\ell) = 0, \tag{3.9}$$

other than the trivial solution $X(x) \equiv 0$.
Case I: Let $\lambda = \beta^2 > 0$. As in the previous subsection, we have

$$X(x) = C\cos\beta x + D\sin\beta x, \quad \text{for } x \in (0,\ell),$$

so that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x, \quad \text{for } x \in (0,\ell).$$

The boundary conditions (3.9) mean first that $0 = X'(0) = D\beta$, so that $D = 0$, and second, that

$$0 = X'(\ell) = -C\beta\sin\beta\ell.$$

Since we do not want $C = 0$, we must have $\sin\beta\ell = 0$. Thus $\beta = \frac{n\pi}{\ell}$ for $n \in \mathbb{N}^+$, Therefore, we have

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad \text{and} \quad X_n(x) = \cos\frac{n\pi x}{\ell}, \quad \text{for } n \in \mathbb{N}^+.$$

Case II. Let $\lambda = 0$. Then $X'' = 0$, so that $X(x) = C + Dx$ and $X'(x) \equiv D$. The Neumann boundary conditions are both satisfied if $D = 0$. C can be any number. Therefore, $\lambda = 0$ is an eigenvalue, and any constant function is its eigenfunctions.

Case III. Let $\lambda < 0$. It can be shown directly, as in the Dirichlet case, that there is no eigenfunction.

In conclusion, the list of all the eigenvalues is

$$\lambda_n = \left(\frac{n\pi}{\ell}\right), \quad \text{for } n \in \mathbb{N}.$$

Note that $n = 0$ is include among them.
So, for instance, the 1D heat equation with the Neumann boundary conditions has the solution

$$u(t,x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 t}\cos\frac{n\pi x}{\ell}.$$

This solution requires the initial data to have the "Fourier cosine expansion"

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\ell}.$$

All the coefficients $\{A_n\}_{n\in\mathbb{N}}$ are just constants.
Consider now the 1D wave equation with the Neumann boundary conditions. The eigenvalue $\lambda = 0$ then leads to $X(x) =$ costant and to the differential equation $T''(t) = \lambda T(t)$, which has the solution $T(t) = A + Bt$. Therefore, the 1D wave equation with Neumann boundary conditions has the solutions

$$u(t,x) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi t}{\ell} + B_n\sin\frac{n\pi t}{\ell}\right)\cos\frac{n\pi x}{\ell}.$$

Then the initial data must satisfy

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\ell},$$

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty}\frac{n\pi}{\ell}B_n\cos\frac{n\pi x}{\ell}.$$

For another example, consider the 1D Schrödinger equation $\partial_t u = i\partial_x^2 u$ in $(0,\ell)$ with the Neumann boundary conditions $\partial_x u(t,0) = \partial_x u(t,\ell) = 0$ and initial condition $u(0,x) = \phi(x)$. Separation of variables leads to the equation

$$\frac{T'}{iT} = \frac{X''}{X} = -\lambda = \text{costant},$$

so that $T(t) = e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (3.9) as before. Therefore, the solution is

$$u(t,x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i\left(\frac{n\pi}{\ell}\right)^2 t} \cos\frac{n\pi x}{\ell}.$$

3.3. **The Robin condition.** We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X'' = \lambda X$ with the boundary conditions

$$\begin{aligned} X' - a_0 X &= 0, \quad \text{at } x = 0, \\ X' + a_\ell X &= 0, \quad \text{at } x = \ell. \end{aligned} \tag{3.10}$$

The two constants a_0 and a_ℓ should be considered as given.

Part I. Positive eigenvalues.

Our goal now is to solve the ODE $-X'' = \lambda X$ with the boundary conditions (3.10). First, let us look for the positive eigenvalues $\lambda = \beta^2 > 0$. As usual, the solution of the ODE is

$$X(x) = C\cos\beta x + D\sin\beta x, \quad \text{for } x \in (0,\ell),$$

so that

$$X'(x) \pm aX(x) = (\beta D \pm aC)\cos\beta x + (-\beta C \pm aD)\sin\beta x.$$

Combing Robin boundary conditions with above identities, we obtain

$$\begin{aligned} 0 &= X'(0) - a_0 X(0) = \beta D - a_0 C, \\ 0 &= (\beta D + a_\ell C)\cos\beta\ell + (-\beta C + a_\ell D)\sin\beta\ell. \end{aligned}$$

Therefore, substituting for D, we have

$$0 = (a_0 + a_\ell)C\cos\beta\ell + \left(-\beta C + \frac{a_0 a_\ell C}{\beta}\right)\sin\beta\ell.$$

We do not want the trivial solution $C \equiv 0$. We divide by $C\cos\beta\ell$ and multiply by β to get

$$(\beta^2 - a_0 a_\ell)\tan\beta\ell = (a_0 + a_\ell)\beta. \tag{3.11}$$

By the way, because we divided by $\cos\beta\ell$, there is the exceptional case when $\cos\beta\ell = 0$; it would mean that $\beta = \sqrt{a_0 a_\ell}$.

Our next goal is to solve (3.10) for β. This is not so easy, as there is no simple formula. One way is to calculate the roots numerically, say by Newton's method. Another way is by graphical analysis, which, instead of precise numerical values, will provide a lot of qualitative information. This is what we will do.

Let us rewrite the eigenvalue equation (3.11) as

$$\tan\beta\ell = \frac{(a_0 + a_\ell\beta)}{\beta^2 - a_0 a_\ell}.$$

Our method is to sketch the graphs of the tangent function $y = \tan\beta\ell$ and the rational function $y = \frac{(a_0+a_\ell)\beta}{\beta^2 - a_0 a_\ell}$ as functions of $\beta > 0$ and to find their points of intersection. What the rational function looks like depends on the constants a_0 and a_ℓ.

Case I. In [4, Page 92, Figure 1] is pictured the case of radiation at both ends: $a_0 > 0$ and $a_\ell > 0$. Each of the points of intersection (for $\beta > 0$) provides an eigenvalue $\lambda_n = \beta_n^2 > 0$. The results depend very much on the a_0 and a_ℓ. The exceptional situation mentioned above, when $\cos\beta\ell = 0$ and $\beta = \sqrt{a_0 a_\ell}$, will occur when the graphs of the tangent function and the rational function "intersect at infinity".

No matter what they are, as long as they are both positive, the graph clearly shows that

$$n^2\frac{\pi^2}{\ell^2} < \lambda_n < (n+1)^2\frac{\pi^2}{\ell^2}, \quad \text{for } n \in \mathbb{N}. \tag{3.12}$$

Furthermore, we have

$$\lim_{n\to\infty}\left(\beta_n - n\frac{\pi}{\ell}\right) = 0, \tag{3.13}$$

which means that the larger eigenvalues get closer and closer to $\frac{n^2\pi^2}{\ell^2}$.

Case II. The case of absorption at $x = 0$ and radiation at $x = \ell$, but more radiation than absorption, is given by the conditions

$$a_0 < 0, \quad a_\ell > 0, \quad a_0 + a_\ell > 0. \tag{3.14}$$

Then the graph look like [4, Page 93, Figure 2 and Figure 3], depending on the relative sizes of a_0 and a_ℓ. Once again we see that (3.12) and (3.13) hold, except that in [4, Page 93, Figure 2] there is no eigenvalue λ_0 in the interval $(0, \frac{\pi^2}{\ell^2})$.

There is an eigenvalue in the interval $(0, \frac{\pi^2}{\ell^2})$ only if the rational curve crosses the first branch of the tangent curve. Since the rational curve has only a single maximum, this crossing can happen only if the slope of the rational curve is greater than the slope of the tangent curve at the origin. Let us calculate these two slopes. A direct calculation shows that the slope $\frac{\mathrm{d}y}{\mathrm{d}\beta}$ of the rational curve at the origin is

$$\frac{a_0 + a_\ell}{-a_0 a_\ell} = \frac{a_\ell - |a_0|}{a_\ell |a_0|} > 0$$

because of (3.14). On the other hand, the slope of the tangent curve $y = \tan\beta\ell$ at the origin is $\ell\sec^2(0) = \ell$. Thus we reach the following conclusion. In case

$$a_0 + a_\ell > -a_0 a_\ell \ell, \tag{3.15}$$

the rational curve will start out at the origin with a greater slope than the tangent curve and the two graphs must intersect at a point in the interval $\left(0, \frac{\pi}{2\ell}\right)$. Therefore, we conclude that in Case 2 there is an eigenvalue $0 < \lambda_0 < \left(\frac{\pi}{2\ell}\right)$ if and only if (3.15) holds.

Part II. Zero Eigenvalue. By an elementary computation, we deduce that there is a zero eigenvalue if and only if

$$a_0 + a_\ell = -a_0 a_\ell \ell. \tag{3.16}$$

Notice that (3.16) can happen only if a_0 and a_ℓ have opposite signs and the interval has exactly a certain length.

Part III. Negative Eigenvalue. Now let us investigate the possibility of a negative eigenvalue. To avoid dealing with imaginary numbers, we set

$$\lambda = -\gamma^2 < 0$$

and write the solution of the differential equation as

$$X(x) = C\cosh\gamma x + D\sinh\gamma x, \quad \text{for } x \in (0, \ell).$$

The boundary conditions, much as before, lead to the eigenvalue equation

$$\tanh\gamma\ell = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}.$$

So we look for intersections of these two graphs for $\gamma > 0$. Any such point of intersection would provide a negative eigenvalue $\lambda = -\gamma^2$ and a corresponding eigenfunction

$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.$$

Several different cases are illustrated in [4, Page 95, Figure 4]. Thus in Case I, of radiation at both ends, when a_0 and a_ℓ are both positive, there is no intersection and so no negative eigenvalue.
Case II, the situation with more radiation than absorption ($a_0 < 0$, $a_\ell > 0$, $a_0 + a_\ell > 0$), is illustrated by the two solid and dashed curves in [4, Page 95, Figure 4]. There is either one intersection or none, depending on the slopes at the origin. The slope of the tanh curve is ℓ, while the slope of the rational curve is $-(a_0 + a_\ell)/(a_0 a_\ell) > 0$. If the last expression is smaller than ℓ, there is an intersection; otherwise, there is not. So our conclusion in Case II is as follows.
Let $a_0 < 0$ and $a_\ell > -a_0$. If

$$a_0 + a_\ell < -a_0 a_\ell \ell, \tag{3.17}$$

then there exists exactly one negative eigenvalue, which we will call $\lambda_0 < 0$. If (3.15) holds, then there is no negative eigenvalue. Notice how the "missing" positive eigenvalue λ_0 in the case (3.17) now makes its appearance as a negative eigenvalue. Furthermore, the zero eigenvalue is the borderline case (3.16); therefore. we use the notation $\lambda_0 = 0$ in the case of (3.16).

Summary. We summarize the various cases as follows:

(i) Case I: only positive eigenvalues.
(ii) Case II with (3.15): only positive eigenvalues.
(iii) Case II with (3.16): Zero is an eigenvalue, all the rest are positive.
(iv) Case II with (3.17): One negative eigenvalue, all the rest are positive.

4. Fourier Series

In this section, we introduce the basic content of Fourier series. A Fourier series is a sum that represents a periodic function as a sum of sine and cosine waves. The frequency of each wave in the sum, or harmonic, is an integer multiple of the periodic function's fundamental frequency. Each harmonic phase and amplitude can be determined using harmonic analysis. A Fourier series may potentially contain an infinite number of harmonics.
In this section, the presentation is close to [3, Chapter 1-3] and [4, Chapter 4].

4.1. **The Coefficients.**

Consider the Fourier sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}, \tag{4.1}$$

in the interval $(0, \ell)$. The first problem we tackle is to try to find the coefficients A_n if $\phi(x)$ is given function. The key observation is that the sine functions have the following wonderful property.

Lemma 4.1. *For all $m, n \in \mathbb{N}$ with $m \neq n$, we have*

$$\int_0^\ell \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} \mathrm{d}x = 0. \tag{4.2}$$

Proof. We use the trig identity,

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b).$$

Therefore, the integral equals

$$\frac{\ell}{2(m-n)\pi}\sin\frac{(m-n)\pi x}{\ell}\bigg|_0^\ell - [\text{same with } (m+n)]\,,$$

if $m \neq n$. This is a linear combination of $\sin(m \pm n)\pi$ and $\sin 0$, and so it vanishes. □

Let us fix m, multiply (4.1) by $\sin(m\pi x/\ell)$ and integrate the series (4.1) term by term to get

$$\begin{aligned}\int_0^\ell \phi(x)\sin\frac{m\pi x}{\ell}\mathrm{d}x &= \int_0^\ell \sum_{n=1}^\infty A_n \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\mathrm{d}x \\ &= \sum_{n=1}^\infty A_n \int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\mathrm{d}x.\end{aligned}$$

All but one term in this sum vanishes, namely the one with $n = m$. Therefore, we are left with the single term

$$A_m \int_0^\ell \sin^2\frac{m\pi x}{\ell}\mathrm{d}x,$$

which equals $\frac{1}{2}\ell A_m$ by explicit integration. Therefore,

$$A_m = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\frac{m\pi x}{\ell}\mathrm{d}x. \tag{4.3}$$

This is the famous formula for the Fourier coefficients in the series (4.1). That is, if $\phi(x)$ has an expansion (4.1), then the coefficients must be given by (4.3).
These are the only possible coefficients in (4.1). However, the basic question still remains whether (4.1) is in fact valid with these values of the coefficients. This is a question of convergence, and we postpone it until Section 4.4.

4.1.1. **Application to the 1D Heat and Wave equation**. Going back to the 1D heat equation with Dirichlet boundary conditions, the formula (4.3) provides the final ingredient in the solution formula for arbitrary initial data $\phi(x)$.
As for the wave equation with Dirichlet conditions, the initial data consist of a pair of functions $(\phi(x), \psi(x))$. The coefficients A_m are given by (4.3), while for the same reason the coefficients B_m are given by the similar formula

$$\frac{m\pi}{\ell}B_m = \frac{2}{\ell}\int_0^\ell \psi(x)\sin\frac{m\pi x}{\ell}\mathrm{d}x.$$

4.1.2. **Fourier cosine series**. Next let us take the case of the cosine series, which corresponds to the Neumann boundary conditions on $(0, \ell)$. We write it as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^\infty A_n \cos\frac{n\pi x}{\ell}.$$

Again we can verify the magical fact that

$$\int_0^\ell \cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\mathrm{d}x = 0, \quad \text{if } m \neq n,$$

where m and n are nonnegative integers. By exactly the same method as above, but with sines replaced by cosines, we get

$$\int_0^\ell \phi(x)\cos\frac{m\pi x}{\ell}\mathrm{d}x = A_m\int_0^\ell \cos^2\frac{m\pi x}{\ell}\mathrm{d}x = \frac{1}{2}\ell A_m,$$

if $m \neq 0$. For the case $m = 0$, we have

$$\int_0^\ell \phi(x) \cdot 1 \mathrm{d}x = \frac{1}{2} A_0 \int_0^\ell 1 \mathrm{d}x = \frac{1}{2} \ell A_0.$$

Therefore, for all nonnegative integers m, we have the formula for the coefficients of the cosine series

$$A_m = \frac{2}{\ell} \int_0^\ell \phi(x) \cos \frac{m\pi x}{\ell} \mathrm{d}x. \tag{4.4}$$

4.1.3. **Full Fourier series**. The full Fourier series, or simply the Fourier series, of $\phi(x)$ on the interval $(-\ell, \ell)$, is defined as

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{\ell} + B_n \sin \frac{n\pi x}{\ell} \right).$$

The interval is twice as long and the eigenfunctions now are all the functions $\{1, \cos(n\pi x/\ell), \sin(n\pi x/\ell)\}$, where $n = 1, 2, 3, \ldots$. Again we have the same wonderful coincidence: Multiply any two different eigenfunctions and integrate over the interval and you will get 0. That is,

$$\begin{aligned}
&\int_{-\ell}^\ell \cos \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} \mathrm{d}x = 0, \quad \text{for all } n, m \in \mathbb{N}^+, \\
&\int_{-\ell}^\ell \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} \mathrm{d}x = 0, \quad \text{for all } n \neq m, \\
&\int_{-\ell}^\ell \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} \mathrm{d}x = 0, \quad \text{for all } n \neq m,
\end{aligned}$$

and

$$\int_{-\ell}^\ell 1 \cdot \cos \frac{n\pi x}{\ell} \mathrm{d}x = \int_{-\ell}^\ell 1 \cdot \sin \frac{n\pi x}{\ell} \mathrm{d}x = 0.$$

Therefore, the same procedure will work to find the coefficients. We also calculate the integrals of the squares

$$\int_{-\ell}^\ell \cos^2 \frac{n\pi x}{\ell} \mathrm{d}x = \int_{-\ell}^\ell \sin^2 \frac{n\pi x}{\ell} \mathrm{d}x = \ell, \quad \int_{-\ell}^\ell 1 \mathrm{d}x = 2\ell.$$

Then we end up with the formulas

$$\begin{aligned}
A_n &= \frac{1}{\ell} \int_{-\ell}^\ell \phi(x) \cos \frac{n\pi x}{\ell} \mathrm{d}x, \quad \text{for } n \in \mathbb{N}^+, \\
B_n &= \frac{1}{\ell} \int_{-\ell}^\ell \phi(x) \sin \frac{n\pi x}{\ell} \mathrm{d}x, \quad \text{for } n \in \mathbb{N}^+,
\end{aligned}$$

for the coefficients of the full Fourier series. Note that these formulas are not exactly the same as (4.3) and (4.4).

Example 4.2. (i) Let $\phi(x) \equiv x$ in the interval $(0, \ell)$. Its Fourier series has the coefficients

$$\begin{aligned}
A_m &= \frac{2}{\ell} \int_0^\ell x \sin \frac{m\pi x}{\ell} \mathrm{d}x \\
&- \frac{2x}{m\pi} \cos \frac{m\pi x}{\ell} + \frac{2\ell}{m^2\pi^2} \sin \frac{m\pi x}{\ell} \Big|_0^\ell \\
&= -\frac{2\ell}{m\pi} \cos m\pi + \frac{2\ell}{m^2\pi^2} \sin m\pi = (-1)^{m+1} \frac{2\ell}{m\pi}.
\end{aligned}$$

Thus in $(0,\ell)$, we have

$$x = \frac{2\ell}{\pi}\left(\sin\frac{\pi x}{\ell} - \frac{1}{2}\sin\frac{2\pi x}{\ell} + \frac{1}{3}\sin\frac{3\pi x}{\ell} - \cdots\right).$$

(ii) Solve the problem

$$\partial_t^2 u = \partial_x^2 u$$
$$u(t,0) = u(t,\ell) = 0,$$
$$u(0,x) = x, \quad \partial_t u(0,x) = 0.$$

From the previous section, we know that $u(t,x)$ has an expansion,

$$u(t,x) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi t}{\ell} + B_n \sin\frac{n\pi t}{\ell}\right)\sin\frac{n\pi x}{\ell}.$$

Differentiating with respect to time yields

$$\partial_t u(t,x) = \sum_{n=1}^{\infty}\frac{n\pi}{\ell}\left(-A_n \sin\frac{n\pi t}{\ell} + B_n \cos\frac{n\pi t}{\ell}\right)\sin\frac{n\pi x}{\ell}.$$

Setting $t=0$, we have

$$0 = \sum_{n=1}^{\infty}\frac{n\pi}{\ell}B_n \sin\frac{n\pi x}{\ell},$$

so that all the $B_n = 0$. Setting $t = 0$ in the expansion of $u(t,x)$, we have

$$x = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}.$$

This is exactly the series of Example 4.2 (i). Therefore, the complete solution is

$$u(t,x) = \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{\ell}\cos\frac{n\pi t}{\ell}.$$

4.2. **Even, Odd, Periodic, and Complex Functions.** Each of the three kinds of Fourier series (sine, cosine and full) of any given function $\phi(x)$ is now determined by the formula for its coefficients given in §4.1. We shall see shortly that almost any function $\phi(x)$ define on the interval $(0,\ell)$ is the sum of its Fourier sine series and is also the sum of its Fourier cosine series. Almost any function defined on the interval $(-\ell,\ell)$ is the sum of its full Fourier series. Each of these series converges inside the interval, but not necessarily at the endpoints.

A function $\phi(x)$ that is defined for $x \in \mathbb{R}$ is called periodic if there is a number $p > 0$ such that

$$\phi(x+p) = \phi(x), \quad \text{for all } x \in \mathbb{R}.$$

The smallest number p for which this is true is called the period of $\phi(x)$. The graph of the function repeats forever horizontally. Note that if $\phi(x)$ has period p, then $\phi(x+np) = \phi(x)$ for all x and for all integers n. The sum of two functions of period p has period p. Notice that if $\phi(x)$ has period p, then $\int_a^{a+p}\phi(x)\mathrm{d}x$ does not depend on a.

If a function is defined only on an interval of length p, it can be extended in only one way to a function of period p. The situation we care about for Fourier series is that of a function defined on the interval $(-\ell,\ell)$. Its periodic extension is

$$\phi_{\text{per}}(x) = \phi(x - 2\ell m), \quad \text{for } x \in (-\ell + 2\ell m, \ell + 2\ell m).$$

This definition does not specify what the periodic extension is at the endpoints $x = \ell + 2\ell m$. In fact, the extension has jumps at these points unless the one-sided limits are equal: $\phi(\ell-) = \phi(\ell+)$.

An even function is a function that satisfies the equation

$$\phi(-x) = \phi(x). \tag{4.5}$$

That just means that its graph $y = \phi(x)$ is symmetric with respect the y axis. Thus the left and right halves of the graph are mirror images of each other. To make sense out of (4.5), we require that $\phi(x)$ be defined on some interval $(-\ell, \ell)$ which is symmetric around $x = 0$.
An odd function is a function that satisfies the equation

$$\phi(-x) = -\phi(x). \tag{4.6}$$

That just means that its graph $y = \phi(x)$ is symmetric with respect the origin. To make sense out of (4.6), we again require that $\phi(x)$ be defined on some interval $(-\ell, \ell)$ which is symmetric around $x = 0$.
It is worth mentioning here that the sum of an even and an odd function can be anything.

Lemma 4.3. *Any function can be written as the sum of an even function and an odd function.*

Proof. Let $f(x)$ be any function at all defined on $(-\ell, \ell)$. Let $\phi(x) = \frac{1}{2}[f(x) + f(-x)]$ and $\psi(x) = \frac{1}{2}[f(x) - f(-x)]$. Then we easily check that $f(x) = \phi(x) + \psi(x)$, that $\phi(x)$ is even and that $\psi(x)$ is odd. The functions $\phi(x)$ and $\psi(x)$ are called the even and odd parts of f, respectively. $\square$

Given any function defined on the interval $(0, \ell)$, it can be extended in only one way to be even or odd. The even extension of $\phi(x)$ is defined as

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < \ell, \\ \phi(-x) & \text{for } -\ell < x < 0. \end{cases}$$

This is just the mirror image. The even extension is not necessarily defined at the origin.
Its odd extension is

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < \ell, \\ -\phi(-x) & \text{for } -\ell < x < 0, \\ 0, & \text{for } x = 0. \end{cases}$$

This is its image through the origin.

4.2.1. **Fourier series and Boundary conditions**. Now let us return to the Fourier sine series. Each of its terms, $\sin(n\pi x/\ell)$, is an odd function. Therefore, its sum (if it converges) also has to be odd. Furthermore, each of its terms has period 2ℓ, so that the same has to be true of its sum. Therefore, the Fourier sine series can be regarded as an expansion of an arbitrary function that is odd and has period 2ℓ defined on the whole line $\mathbb{R}$.
Similarly, since all the cosine functions are even, the Fourier cosine series can be regarded as an expansion of an arbitrary function which is even and has period 2ℓ defined on the whole line $\mathbb{R}$.
From what we saw in §4.1, these concepts therefore have the following relationship to boundary conditions:

(i) $u(t, 0) = u(t, \ell) = 0$: Dirichlet Boundary conditions correspond to the odd extension.
(ii) $\partial_x u(t, 0) = \partial_x u(t, \ell) = 0$: Neumann Boundary conditions correspond to the even extension.
(iii) $u(t, \ell) = u(t, -\ell)$, $\partial_x u(t, \ell) = \partial_x u(t, -\ell)$: Periodic Boundary conditions correspond to the periodic extension.

4.2.2. **The complex form of the full Fourier series**. The eigenfunctions of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $(-\ell, \ell)$ with the periodic boundary conditions are $\sin(n\pi x/\ell)$ and $\cos(n\pi x/\ell)$. But recall the DeMoivre formulas, which express the sine and cosine in terms of the complex exponentials:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Therefore, instead of sine and cosine, we could use $\left\{e^{in\pi x/\ell}\right\}_{n\in\mathbb{Z}}$ as an alternative pair.

We should therefore be able to write the full Fourier series in the complex form

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}.$$

This is the sum of two infinite series, one going from $n = 0$ to $+\infty$ and one going from $n = -1$ to $-\infty$. Note that, for $m \neq n$, we have

$$\int_{-\ell}^{\ell} e^{im\pi x/\ell} e^{-in\pi x/\ell} \mathrm{d}x = \int_{-\ell}^{\ell} e^{i(m-n)\pi x/\ell} = 0.$$

When $m = n$, we have

$$\int_{-\ell}^{\ell} e^{i(m-n)\pi x/\ell} \mathrm{d}x = \int_{-\ell}^{\ell} 1 \mathrm{d}x = 2\ell.$$

It follows by the method of §4.1 that the coefficients are given by the formula

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} \phi(x) e^{-in\pi x/\ell} \mathrm{d}x.$$

4.3. **Orthogonality and General Fourier series.** For $f(x), g(x) \in C([a, b] : \mathbb{R})$, we define their inner product to be the integral of their product:

$$(f, g) = \int_a^b f(x) g(x) \mathrm{d}x.$$

We will call $f(x)$ and $g(x)$ orthogonal if $(f, g) = 0$. Notice that no function is orthogonal to itself except $f(x) \equiv 0$. The key observation in each case discussed in §4.1 is that every eigenfunctions is orthogonal to every other eigenfunction. Now we will explain why this fortuitous coincidence is in fact no accident.

Consider the operator $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with some boundary conditions. Let $X_1(x)$ and $X_2(x)$ be two different eigenfunctions. Thus

$$-X_1'' = \lambda_1 X_1 \quad \text{and} \quad -X_2'' = \lambda_2 X_2, \tag{4.7}$$

where both functions satisfy the boundary conditions. Let us assume that $\lambda_1 \neq \lambda_2$. We now verify the identity

$$-X_1'' X_2 + X_1 X_2'' = \left(-X_1' X_2 + X_1 X_2'\right)'.$$

Integrating the above identity on $[a, b]$, we obtain

$$\int_a^b \left(-X_1'' X_2 + X_1 X_2''\right) \mathrm{d}x = \left(-X_1' X_2 + X_1 X_2'\right)\Big|_a^b. \tag{4.8}$$

On the left-hand side of (4.8), we now use the differential equation (4.7). On the right-hand side, we use the boundary conditions to reach the following conclusions:

(i) Dirichlet. This means that both functions vanish at both ends: $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$. So the right-hand side of (4.8) is 0.
(ii) Neumann. The first derivatives vanish at both ends. It is once again 0.
(iii) Periodic. $X_j(a) = X_j(b)$, $X_j'(a) = X_j'(b)$ for $j = 1, 2$, and so we have 0.
(iv) Robin. By a similar argument, we also get 0.

Thus in all four cases, (4.8) reduces to

$$(\lambda_1 - \lambda_2) \int_a^b X_1 X_2 \mathrm{d}x = 0,$$

which means that X_1 and X_2 are orthogonal at least for $\lambda_1 \neq \lambda_2$.

4.3.1. **Symmetric Boundary conditions**. So now let us consider any pair of boundary conditions

$$\begin{aligned} \lambda_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0, \\ \lambda_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0, \end{aligned} \tag{4.9}$$

involving eight real constants. Such a set of boundary conditions is called symmetric if

$$f'(x)g(x) - f(x)g'(x)|_a^b = 0,$$

for any pair of functions $f(x)$ and $g(x)$ both of which satisfy the pair of boundary conditions (4.9). As we indicated above, each of the four standard boundary conditions is symmetric. The identity (4.8) then implies the following theorem.

Theorem 4.4. *If you have symmetric boundary conditions, then any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. Therefore, if any function is expanded in a series of these eigenfunctions, the coefficients are determined.*

Proof. Take two different eigenfunctions $X_1(x)$ and $X_2(x)$ with $\lambda_1 \neq \lambda_2$. We write the identity (4.8). Because the boundary conditions are symmetric, the right-hand side of (4.8) vanishes which implies the orthogonality.
If $X_n(x)$ now denotes the eigenfunction with eigenvalue λ_n and if

$$\phi(x) = \sum_n A_n X_n(x),$$

is a convergent series, where the A_n are constants, then

$$(\phi, X_m) = \left(\sum_n A_n X_n(x), X_m \right) = \sum_n A_n (X_n, X_m) = A_m (X_m, X_m),$$

by the orthogonality. So if we denote $c_m = (X_m, X_m)$, we have

$$A_m = c_m^{-1} (\phi, X_m),$$

as the formula for the coefficients. □

Two words of caution. First, we have so far avoided all questions of convergence. Second, if there are two eigenfunctions, say $X_1(x)$ and $X_2(x)$, but their eigenvalues are the same $\lambda_1 = \lambda_2$, then they do not have to be orthogonal. But if they are not orthogonal, they can be made so by the Gram-Schmidt orthogonalization procedure.

4.3.2. **Complex Eigenvalues**. Let $f(x)$ and $g(x)$ be two complex-valued functions, we define the inner product on (a, b) as

$$(f, g) = \int_a^b f(x)\bar{g}(x)\mathrm{d}x.$$

The bar denotes the complex conjugate. The two functions are called orthogonal if $(f, g) = 0$.
Now suppose that you have the boundary conditions (4.9) with eight real constants. They are called symmetric if

$$(f'(x)\bar{g}(x) - f(x)\bar{g}'(x))|_a^b = 0,$$

for all f, g satisfying the boundary conditions. Then Theorem 4.4 is also true for complex functions without any change at all. But we also have the following important fact.

Theorem 4.5. *Under the same conditions as Theorem 4.4, all the eigenvalues are real. Furthermore, all the eigenfunctions can be chosen to be real valued.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue. Let $X(x) : [a, b] \to \mathbb{C}$ be its eigenfunctions. Then we have $-X'' = \lambda X$ plus the boundary conditions. Take the complex conjugate of this equation; thus $-\bar{X}'' = \bar{\lambda}\bar{X}$ plus the boundary conditions. So $\bar{\lambda}$ is also an eigenvalue. Using again (4.8), we have

$$\int_a^b \left(-X''\bar{X} + X\bar{X}''\right) \mathrm{d}x = \left(-X'\bar{X} + X\bar{X}'\right)\big|_a^b = 0,$$

since the boundary conditions are symmetric. So we have

$$\left(\lambda - \bar{\lambda}\right) \int_a^b X\bar{X}\mathrm{d}x = \left(\lambda - \bar{\lambda}\right) \int_a^b |X|^2 \mathrm{d}x = 0.$$

Based on the above identity, we have $\lambda - \bar{\lambda} = 0$ which implies $\lambda \in \mathbb{R}$.
Next, let us consider again the same eigenvalue problem $-X'' = \lambda X$ together with (4.9), knowing that λ is real. If X is complex, we write it as $X(x) = Y(x) + iZ(x)$, where $Y(x)$ and $Z(x)$ are real. Then we have $-Y'' - iZ'' = \lambda Y + i\lambda Z$. Equating the real and imaginary parts, we see that

$$-Y'' = \lambda Y \quad \text{and} \quad -Z'' = \lambda Z.$$

The boundary conditions still hold for both Y and Z because the eight constants in (4.9) are real numbers. So the real eigenvalue λ has the real eigenfunctions Y and Z. We could therefore say that X and $\bar{X}$ are replaceable by the Y and Z. The linear combinations $aX + b\bar{X}$ are the same as the linear combinations $cY + dZ$ where $a, b, c, d \in \mathbb{C}$. The proof of Theorem 4.5 is complete. □

4.3.3. **Negative Eigenvalues**. We have seen that most of the eigenvalues turn out to be positive. An important question is whether all of them are positive. Here is a sufficient condition.

Theorem 4.6. *Assume the same conditions as in Theorem 4.4. If*

$$f(x)f'(x)\big|_{x=a}^{x=b} \leq 0, \tag{4.10}$$

for all real-valued functions $f(x)$ satisfying the boundary conditions, then there is no negative eigenvalue.

Proof. Let $\lambda \in \mathbb{R}$ be an eigenvalue and $f(x)$ be its eigenfunction:

$$-X'' = \lambda X \Rightarrow -(XX')' + (X')^2 = \lambda X^2.$$

Integrating the above identity and using (4.10), we have

$$\lambda \int_a^b (X)^2 \mathrm{d}x = -\left(XX'\right)\big|_a^b + \int_a^b (X)^2 \mathrm{d}x \geq 0,$$

which implies $\lambda \geq 0$. □

4.4. **Completeness.** In this section, we state the basic Theorem about the convergence of Fourier series. We discuss three senses of convergence of functions. The basic Theorems state sufficient conditions on a function $f(x)$ that its Fourier series converge to it in these three senses.
Consider the eigenvalue problem

$$-X'' = \lambda X \quad \text{in } (a, b) \text{ with any symmetric boundary conditions.} \tag{4.11}$$

By Theorem 4.5, we know that all the eigenvalues λ are real.

Theorem 4.7. *There are an infinite number of eigenvalues. They form a sequence $\lambda_n \to \infty$.*

For a proof of Theorem 4.7, we refer to [4, Chapter 11]. We may assume that the eigenvalues series $\{\lambda_n\}_{n=1}^{\infty}$ is increasing and the eigenfunctions $X_n(x)$ are pairwise orthogonal and real valued.
For any function $f(x)$ on (a,b), its Fourier coefficients are defined as

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{\int_a^b f(x)\overline{X}_n(x)\mathrm{d}x}{\int_a^b |X_n(x)|^2\mathrm{d}x}.$$

Its Fourier series is the series $\sum_n A_n X_n(x)$.
In this section, we present three convergence theorem. To set the stage, we need to introduce various notions of convergence.

4.4.1. **Three notions of convergence.**

Definition 4.8. We say that an infinite series $\sum\limits_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ pointwise in (a,b) if it converges to $f(x)$ for each $x \in (a,b)$. That is, for each $x \in (a,b)$, we have

$$\lim_{N\to\infty} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| = 0.$$

Definition 4.9. We say that an infinite series $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly to $f(x)$ in $[a,b]$ if

$$\lim_{N\to\infty} \max_{x\in[a,b]} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| = 0.$$

That is, you take the biggest difference over all the x's and then take the limit. A third important concept is the following one.

Definition 4.10. We say the series converges in the mean-square sense to $f(x)$ in (a,b) if

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \mathrm{d}x = 0.$$

Notice that uniform convergence is stronger than both pointwise and L^2 convergence.

Example 4.11. (i) Let $f_n(x) = (1-x)x^{n-1}$ on the interval $x \in (0,1)$. By an elementary computation,

$$\sum_{n=1}^{N} f_n(x) = 1 - x^N \to 1, \quad \text{as } N \to \infty.$$

This convergence is valid for each $x \in (0,1)$. Thus $\sum\limits_{n=1}^{\infty} f_n(x) = 1$ pointwise which means that the series converges pointwise to the function $f(x) \equiv 1$.
But the convergence is not uniform because $\sup\limits_{x\in(0,1)} \left[1 - (1 - x^N)\right] = 1$ for every N.
However, it does converge in mean-square since

$$\int_0^1 x^{2N}\mathrm{d}x = \frac{1}{2N+1} \to 0.$$

(ii) Let

$$f_n(x) = \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2}, \quad \text{in the interval } (0,\ell).$$

This series also telescopes so that

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1+N^2x^2} = \frac{1}{N\left[(1/N)^2+x^2\right]} \to 0, \quad \text{as } N \to \infty.$$

So the series converges pointwise to the sum $f(x) \equiv 0$.
On the other hand,

$$\begin{aligned}\int_0^\ell \left[\sum_{n=1}^{N} f_n(x)\right]^2 \mathrm{d}x &= \int_0^\ell \frac{N^2}{(1+N^2x^2)^2}\mathrm{d}x \\ &= N\int_0^{N\ell} \frac{1}{(1+y^2)^2}\mathrm{d}y \to \infty, \quad \text{as } N \to \infty.\end{aligned}$$

So the series does not converge in the mean-square sense. Also, it does not converge uniformly because

$$\max_{x\in[0,\ell]} \frac{N}{1+N^2x^2} = N,$$

which obviously does not tend to zero as $N \to \infty$.

4.4.2. **Convergence Theorems**. Now let $f(x)$ be any function defined on $x \in [a,b]$. Consider the Fourier series for the problem (4.11) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 4.12 (Uniform Convergence)**.** *The Fourier series $\sum A_n X_n(x)$ converges to $f(x)$ uniformly on $[a,b]$ provided that*

(i) *$f(x)$ and $f'(x)$ exist and are continuous for $x \in [a,b]$,*
(ii) *$f(x)$ satisfies the given boundary conditions.*

Theorem 4.12 assures us of a very good kind of convergence provided that the conditions on $f(x)$ and its derivatives are met.

Theorem 4.13 (L^2 convergence)**.** *The Fourier series converges to $f(x)$ in the mean-square sense in (a,b) provided only that $f(x)$ is a continuous function.*

Theorem 4.14. (Pointwise Convergence of Classical Fourier Series).

(i) *The classical Fourier series (sine or cosine or full) converges to $f(x)$ pointwise on (a,b), provided that $f(x)$ is a continuous function on $[a,b]$ and $f'(x)$ is piecewise continuous on $[a,b]$.*
(ii) *More generally, if $f(x)$ itself is only piecewise continuous on $[a,b]$ and $f'(x)$ is also piecewise continuous on $[a,b]$, then the classical Fourier series converges at every point x. The sum is*

$$\sum_n A_n X_n(x) = \frac{1}{2}\left(f(x+)+f(x-)\right), \quad \text{for all } x \in (a,b).$$

4.4.3. **The L^2 theory**. We have already defined the inner product on (a,b) as

$$(f,g) = \int_a^b f(x)\bar{g}(x)\mathrm{d}x.$$

We now define the L^2 norm of f as

$$\|f\| = (f,f,)^{\frac{1}{2}} = \left(\int_a^b |f(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

The quantity

$$\|f-g\| = \left(\int_a^b |f(x)-g(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}},$$

is a measurement of the distance between two functions f and g. It is some times called the L^2 metric.
Theorem 4.13 can be restated as follows. If $\{X_n\}$ are the eigenfunctions associated with a set of symmetric boundary conditions and if $\|f\| < \infty$, then

$$\left\| f - \sum_{n=1}^{N} A_n X_n \right\| \to 0, \quad \text{as } N \to \infty.$$

Lemma 4.15 (Best Approximation)**.** *Let $\{X_n\}$ be any orthogonal set of functions and $\|f\| < \infty$. Let N be a fixed positive integer. Among all possible choices of N constants $\{c_n\}_{n=1}^N$, the choice that minimizes*

$$\left\| f - \sum_{n=1}^{N} c_n X_n \right\|$$

is $c_1 = A_1, \cdots, c_N = A_N$.

Proof. For the sake of simplicity, we assume in this proof that $f(x)$ and all the $X_n(x)$ are real valued. Denote the error by

$$E_N = \left\| f - \sum_{n=1}^{N} c_n X_n \right\|^2 = \int_a^b \left| f(x) - \sum_{n=1}^{N} c_n X_n(x) \right|^2 \mathrm{d}x.$$

Expanding the square, we have

$$\begin{aligned} E_N = &\int_a^b |f(x)|^2 \mathrm{d}x - 2\sum_{n=1}^{N} c_n \int_a^b f(x) X_n(x) \mathrm{d}x \\ &+ \sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m \int_a^b X_n(x) X_m(x) \mathrm{d}x. \end{aligned}$$

Because of orthogonality, the last integral vanished except for $n = m$, and so

$$\begin{aligned} E_N &= \|f\|^2 - 2\sum_{n=1}^{N} c_n (f, X_n) + \sum_{n=1}^{N} c_n^2 \|X_n\|^2 \\ &= \sum_{n=1}^{N} \|X_n\|^2 \left(c_n - \frac{(f, X_n)}{\|X_n\|^2} \right)^2 + \|f\|^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|^2}. \end{aligned}$$

Now the coefficients c_n appear in only one place, inside the squared term. The expression is clearly smallest if the squared term vanished. That is,

$$c_n = \frac{(f, X_n)}{\|X_n\|^2} = A_n.$$

□

Note that, from $E_N \geq 0$, we also have

$$0 \leq E_N \leq \|f\|^2 - \sum_{n=1}^{N} A_n^2 \|X_n\|^2, \tag{4.12}$$

which implies

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 \mathrm{d}x \leq \int_a^b |f(x)|^2 \mathrm{d}x.$$

This is known as Bessel's inequality. It is valid as long as the integral of $|f|^2$ is finite.

Theorem 4.16. *The Fourier series of $f(x)$ converges to $f(x)$ in the mean-square sense if and only if*

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n|^2 \mathrm{d}x = \int_a^b |f(x)|^2 \mathrm{d}x. \tag{4.13}$$

Proof. Mean-square convergence means that the remainder term $E_N \to 0$ as $N \to \infty$. But from (4.12) this means that

$$\lim_{N\to\infty} \sum_{n=1}^{N} A_n^2 \|X_n\|^2 = \|f\|^2,$$

which in turn means (4.13), known as Parseval's equality. □

Definition 4.17. The infinite orthogonal set of functions $\{X_n\}_{n=1}^{\infty}$ is called complete if Parseval's equality (4.12) is true for all f with $\int_a^b |f(x)|^2 \mathrm{d}x < \infty$.

Last, we give the proof that the L^2 convergence of full Fourier series using the best approximation Lemma, as well as the important fact that trigonometric polynomials are dense in the space of continuous functions on $[-\pi, \pi]$. Moreover, we conclude that the full Fourier basic $\left\{e^{inx}\right\}_{n\in Z}$ or $\{1, \sin nx, \cos nx\}_{n\in\mathbb{N}^+}$ is complete on $[-\pi, \pi]$.

Proof of Theorem 4.13. Without loss of generality, we suppose that $f(x)$ is continuous on $[-\pi, \pi]$. Then, given $\varepsilon > 0$, there exists a trigonometric polynomial P, say of degree N such that

$$\max_{x\in[-\pi,\pi]} |f(x) - P(x)| < \varepsilon.$$

In particular, taking squares and integrating this inequality yields

$$\int_{-\pi}^{\pi} |f(x) - P(x)|^2 \mathrm{d}x \le 2\pi\varepsilon^2,$$

and by the best approximation lemma we conclude that

$$\int_{-\pi}^{\pi} \left| f(x) - \sum_{n=1}^{N} A_n X_n(x) \right|^2 \mathrm{d}x \le \int_{-\pi}^{\pi} |f(x) - P(x)|^2 \mathrm{d}x \le 2\pi\varepsilon^2.$$

This proves Theorem 4.13 when f is continuous. □

4.5. **Completeness and the Gibbs phenomenon.** Our purpose here is to prove the pointwise convergence of the classical full Fourier series. We assume that $\ell = \pi$, which can easily be arranged through a change of scale.
Thus the Fourier series is

$$f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx\right),$$

with the coefficients

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \mathrm{d}y, \quad \text{for } n \ge 0,$$
$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \mathrm{d}y, \quad \text{for } n \ge 1.$$

The Nth partial sum of the series is

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}\left(A_n \cos nx + B_n \sin nx\right).$$

We want to prove that $S_N(x)$ converges to $f(x)$ as $N \to \infty$. Pointwise convergence means that x is kept fixed as we take the limit.
Using the fact that $\cos a \cos b + \sin a \sin b = \cos(a-b)$, the partial sum $S_N(x)$ can be rewritten as

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\mathrm{d}y,$$

where

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta.$$

Now we study the property of this function K_N, called the Dirichlet kernel. Note that $K_N(\theta)$ has period 2π and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\mathrm{d}\theta + \frac{1}{2\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi}\cos n\theta \mathrm{d}\theta = 1.$$

Moreover, we have the following remarkable fact.

Lemma 4.18. *For all $N \in \mathbb{N}^+$, we have*

$$K_N(\theta) = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}. \tag{4.14}$$

Proof. Using the fact that $\cos n\theta = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right)$, we have

$$K_N(\theta) = 1 + \sum_{n=1}^{N}\left(e^{in\theta} + e^{-in\theta}\right) = \sum_{n=-N}^{N} e^{in\theta}.$$

In actually, this is a finite geometric series with the first term $e^{-iN\theta}$, the radio $e^{i\theta}$ and the last term $e^{iN\theta}$. Therefore, we have

$$\begin{aligned} K_N(\theta) &= \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} \\ &= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}} = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}. \end{aligned}$$

□

4.5.1. **Proof for C^1 functions**. By a change of variable, we have

$$\begin{aligned} S_N(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\left(f(x+\theta) - f(x)\right)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\mathrm{d}\theta, \end{aligned}$$

where

$$g(\theta) = \frac{f(x+\theta) - f(x)}{\sin\frac{1}{2}\theta}, \quad \text{for } \theta \in [-\pi, \pi].$$

By an elementary computation, we see that $\left\{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\right\}_{N\in\mathbb{N}^+}$ form an orthogonal set on the interval $(-\pi,\pi)$ and

$$\left\|\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\right\|^2 = \pi, \quad \text{for all } N \in \mathbb{N}^+.$$

On the other hand, using L'Hôpital's rule, we have

$$\lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0}\left(\frac{f(x+\theta)-f(x)}{\theta}\cdot\frac{\theta}{\sin\frac{\theta}{2}}\right) = 2f'(x).$$

Therefore, $g(\theta)$ is everywhere continuous, so that the integral $\|g\|$ is finite. Combining the above identity and inequality with the Bessel's inequality, we have

$$\sum_{n=1}^{\infty}\left|\left(g(\theta), \sin\left(\left(N+\frac{1}{2}\right)\theta\right)\right)\right|^2 < \infty \Rightarrow \lim_{N\to\infty}\left|\left(g(\theta), \sin\left(\left(N+\frac{1}{2}\right)\theta\right)\right)\right| = 0,$$

which means that

$$\lim_{N\to\infty}|S_N(x)-f(x)| = 0, \quad \text{for all } x\in[-\pi,\pi].$$

This completes the proof of pointwise convergence of the Fourier series of any C^1 function.

4.5.2. **Proof for discontinuous functions**. If the periodic function $f(x)$ itself is only piecewise continuous and $f'(x)$ is also piecewise continuous on (a,b), we want to prove that the Fourier series converges and that its sum is $\frac{1}{2}[f(x+)+f(x-)]$. This means that we assume that $f(x)$ and $f'(x)$ are continuous except at a finite number of points, and at those points they have jump discontinuities.
The strategy of the proof is similar to before. Note that

$$\begin{aligned}
&S_N(x) - \frac{1}{2}\left(f(x+)+f(x-)\right)\\
&= \frac{1}{2\pi}\int_0^{\pi} K_N(\theta)\left(f(x+\theta)-f(x+)\right)\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{0} K_N(\theta)\left(f(x+\theta)-f(x-)\right)\mathrm{d}\theta\\
&= \frac{1}{2\pi}\int_0^{\pi} g_+(\theta)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{0} g_-(\theta)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\mathrm{d}\theta,
\end{aligned}$$

where

$$g_+(\theta) = \frac{f(x+\theta)-f(x+)}{\sin\frac{1}{2}\theta} \quad\text{and}\quad g_-(\theta) = \frac{f(x+\theta)-f(x-)}{\sin\frac{1}{2}\theta}.$$

Note that, from the Bessel's inequality, and f and f' are piesewise continuous, we have

$$\sum_{\pm}\|g_+\|^2 < \infty \Rightarrow \lim_{N\to\infty}\sum_{\pm}\left|\left(g_\pm, \sin\left(\left(N+\frac{1}{2}\right)\theta\right)\right)\right| = 0,$$

which means that $S_N(x)$ converges to $\frac{1}{2}\left(f(x+)+f(x-)\right)$.

4.5.3. **Proof of uniform convergence**. Now we prove Theorem 4.12. The idea is to show that the coefficients go to zero pretty fast by integration by parts. More precisely, let A_n and B_n be the Fourier coefficients of $f(x)$ and let A_n' and B_n' denote the Fourier coefficients of $f'(x)$. By integration by parts, we have

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x\\
&= \frac{1}{n\pi}f(x)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\mathrm{d}x = -\frac{1}{n}B_n', \quad \text{for } n\neq 0.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\mathrm{d}x\\
&= \frac{1}{n\pi}f(x)\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\mathrm{d}x = \frac{1}{n}A_n', \quad \text{for } n\neq 0.
\end{aligned}$$

On the other hand, from the Bessel's inequality, we know that

$$\|f'\|^2 < \infty \Rightarrow \sum_{n=1}^{\infty} \left(|A_n'|^2 + |B_n'|^2\right) < \infty.$$

Therefore, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} (|A_n| + |B_n|) &\le \sum_{n=1}^{\infty} \frac{1}{n} (|A_n'| + |B_n'|) \\ &\le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left(|A_n'|^2 + |B_n'|^2\right)\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

Based on the above inequality, we see that the Fourier series converges absolutely. Moreover, we have

$$\begin{aligned}\max_{x\in[-\pi,\pi]} |S_N(x) - f(x)| &\le \max_{x\in[-\pi,\pi]} \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx| \\ &\le \sum_{n=N+1}^{\infty} (|A_n| + |B_n|) \to 0, \quad \text{as } N \to \infty,\end{aligned}$$

which means that the Fourier series converges uniformly to $f(x)$.

4.5.4. **The Gibbs phenomenon**. The Gibbs phenomenon is what happens to Fourier series at jump discontinuities. More precisely, the Gibbs phenomenon states that near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately 9% of the jump.

We now verify the Gibbs phenomenon for an example. Consider the following odd function,

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{for } 0 < x < \pi, \\ -\dfrac{1}{2} & \text{for } -\pi < x < 0. \end{cases}$$

By an elementary computation, the function f has the following Fourier series

$$\sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin((2n-1)\pi).$$

Recall that, the partial sum $S_N(x)$ is given by

$$\begin{aligned}S_N(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-y) f(y) \mathrm{d}y \\ &= \frac{1}{4\pi} \int_0^{\pi} \frac{\sin\left((N+\frac{1}{2})(x-y)\right)}{\sin\frac{1}{2}(x-y)} \mathrm{d}y - \frac{1}{4\pi} \int_{-\pi}^{0} \frac{\sin\left((N+\frac{1}{2})(x-y)\right)}{\sin\frac{1}{2}(x-y)} \mathrm{d}y.\end{aligned}$$

Let $M = N + \frac{1}{2}$. Consider the change of variable $\theta = M(x-y)$ in the first integral and the change of variable $\theta = M(y-x)$ in the second integral,

$$\begin{aligned}S_N(x) &= \frac{1}{2\pi} \int_{M(x-\pi)}^{Mx} \frac{\sin\theta}{2M\sin(\theta/2M)} \mathrm{d}\theta - \frac{1}{2\pi} \int_{-M(x+\pi)}^{-Mx} \frac{\sin\theta}{2M\sin(\theta/2M)} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-Mx}^{Mx} \frac{\sin\theta}{2M\sin(\theta/2M)} \mathrm{d}\theta - \frac{1}{2\pi} \int_{M(\pi-x)}^{M(\pi+x)} \frac{\sin\theta}{2M\sin(\theta/2M)} \mathrm{d}\theta.\end{aligned}$$

Let $x = \frac{\pi}{M}$ in the above identity, we get

$$S_N\left(\frac{\pi}{M}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\theta}{2M\sin(\theta/2M)} \mathrm{d}\theta - \frac{1}{2\pi} \int_{(M-1)\pi}^{(M+1)\pi} \frac{\sin\theta}{2M\sin(\theta/2M)} \mathrm{d}\theta.$$

On the one hand, we have

$$
\begin{aligned}
&2M\sin\frac{\theta}{2M} \to \theta \text{ uniformly in } \theta\in[-\pi,\pi]\\
\Rightarrow\ &\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\theta}{2M\sin(\theta/2M)}\mathrm{d}\theta \to \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\theta}{\theta}\mathrm{d}\theta, \quad \text{as } M\to\infty.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\theta}{2M}\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right], \quad \text{for all } x\in[(M-1)\pi,(M+1)\pi]\\
\Rightarrow\ &\left|\frac{1}{2\pi}\int_{(M-1)\pi}^{(M+1)\pi}\frac{\sin\theta}{2M\sin(\theta/2M)}\mathrm{d}\theta\right| \le \frac{1}{\sqrt{2}M}\to 0 \quad \text{as } M\to\infty.
\end{aligned}
$$

Combining the above two estimates, we have

$$
\lim_{M\to\infty} S_N\left(\frac{\pi}{M}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\theta}{\theta}\mathrm{d}\theta \simeq 0.59,
$$

which implies

$$
\left|S_N\left(\frac{\pi}{M}\right)-f(x)\right| = \left|S_N\left(\frac{\pi}{M}\right)-\frac{1}{2}\right| = 0.09.
$$

4.6. **Inhomogeneous Boundary conditions**. In this subsection, we consider problems with sources given at the boundary.

4.6.1. **1D heat equation**. Consider the 1D heat equation with sources at both endpoints

$$
\begin{aligned}
&\partial_t u = \partial_x^2 u, \quad \text{for } (t,x)\in(0,\infty)\times(0,\pi),\\
&(u(t,0),u(t,\pi),u(0,x)) = (h(t),j(t),0).
\end{aligned} \tag{4.15}
$$

For each $t\in(0,\infty)$, we certainly can expand

$$
u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin nx.
$$

The coefficients are necessarily given by

$$
u_n(t) = \frac{2}{\pi}\int_0^{\pi} u(t,x)\sin nx\mathrm{d}x.
$$

Form the initial data $u(0,x)=0$ for all $x\in[0,\pi]$, we know that $u_n(0)=0$ for all $n\in\mathbb{N}^+$. For the term $\partial_t u$, we also expand it by Fourier series,

$$
\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} v_n(t)\sin nx \quad \text{with } v_n(t) = \frac{2}{\pi}\int_0^{\pi}\frac{\partial u}{\partial t}\sin nx\mathrm{d}x = \frac{\mathrm{d}u_n}{\mathrm{d}t}.
$$

For the term $\partial_x^2 u$, we also expand it by Fourier series,

$$
\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} w_n(t)\sin nx \quad \text{with } w_n(t) = \frac{2}{\pi}\int_0^{\pi}\frac{\partial^2 u}{\partial x^2}\sin nx\mathrm{d}x.
$$

Moreover, by integration by parts, we have

$$
\begin{aligned}
w_n(t) &= -\frac{2n^2}{\pi}\int_0^{\pi} u(t,x)\sin nx\mathrm{d}x - \frac{2n}{\pi}\, u(t,x)\cos nx|_0^{\pi}\\
&= -n^2 u_n(t,x) + \frac{2n}{\pi}\left[(-1)^{n+1}j(t)+h(t)\right].
\end{aligned}
$$

Moreover, using the fact that $\partial_t u = \partial_x^2 u$, we have

$$
v_n(t) = w_n(t) \Rightarrow \frac{\mathrm{d}u_n}{\mathrm{d}t} = -n^2 u_n(t,x) + \frac{2n}{\pi}\left[(-1)^{n+1}j(t)+h(t)\right].
$$

Based on the above identity and $u_n(0) = 0$, we have

$$u_n(t) = \frac{2n}{\pi} \int_0^t e^{-n^2(t-s)} \left[(-1)^{n+1} j(s) + h(s)\right] \mathrm{d}s, \quad \text{for } n \in \mathbb{N}^+.$$

4.6.2. **1D wave equation**. Consider the following 1D inhomogeneous wave equation

$$\begin{gathered} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = f(t,x), \\ u(t,0) = h(t) \quad \text{and} \quad u(t,\pi) = k(t), \\ u(0,x) = \phi(x) \quad \text{and} \quad \partial_t u(0,x) = \psi(x). \end{gathered}$$

Again we expand everything in the eigenfunctions of the corresponding homogeneous problem:

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t) \sin nx,$$

$\partial_t^2 u(t,x)$ with coefficients $v_n(t)$, $\partial_x^2 u(t,x)$ with $w_n(t)$, $f(t,x)$ with coefficients $f_n(t)$, $\phi(x)$ with coefficients ϕ_n and $\psi(x)$ with coefficients ψ_n. Then, from the definition of Fourier coefficients, we have

$$v_n(t) = \frac{2}{\pi} \int_0^\pi \frac{\partial^2 u}{\partial t^2} \sin nx \mathrm{d}x = \frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2},$$

and just as before,

$$\begin{aligned} w_n(t) &= \frac{2}{\pi} \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx \mathrm{d}x \\ &= -n^2 u_n(t) + \frac{2n}{\pi} \left(h(t) + (-1)^{n+1} k(t)\right). \end{aligned}$$

From the PDE $\partial_t^2 u - \partial_x^2 u = f$, we have

$$v_n(t) - w_n(t) = \frac{2}{\pi} \int_0^\pi (\partial_t^2 u - \partial_x^2 u) \sin nx \mathrm{d}x = f_n(t).$$

Based on the above identity, we have the following second-order ODE

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} + n^2 u_n = \frac{2n}{\pi} \left(h(t) + (-1)^{n+1} k(t)\right) + f_n(t),$$

with the initial conditions

$$u_n(0) = \phi_n \quad \text{and} \quad \partial_t u_n(0) = \psi_n.$$

The solution can be given by

$$\begin{aligned} u_n(t) &= \phi_n \cos nt + \frac{1}{n} \psi_n \sin nt \\ &\quad + \frac{1}{n} \int_0^t \sin(n(t-s)) \left(\frac{2n}{\pi} \left(h(s) + (-1)^{n+1} k(s)\right) + f_n(s) \right) \mathrm{d}s. \end{aligned}$$

References

[1] L. C. Evans. Partial differential equations. Second edition. Graduate Studies in Mathematics, **19**. *American Mathematical Society, Providence, RI,* 2010.

[2] F. John. Partial differential equations. Reprint of the fourth edition. Applied Mathematical Sciences, 1. *Springer-Verlag, New York,* 1991. x+249 pp.

[3] E. M. Stein and R. Shakarchi. Fourier analysis. An introduction. Princeton Lectures in Analysis, 1. *Princeton University Press, Princeton, NJ,* 2003. xvi+311 pp.

[4] W. Strauss. Partial differential equations. An introduction. *John Wiley & Sons, Inc., New York,* 1992. xii+425 pp.

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